# LINEAR BETA PRICING WITH INEFFICIENT BENCHMARKS

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**Abstract.** Current asset pricing models require mean-variance efficient benchmarks, which are generally unavailable because of partial securitization and free float restrictions. We provide a pricing model that uses inefficient benchmarks, a two-beta model, one induced by the benchmark and one adjusting for its inefficiency. While efficient benchmarks induce zero-beta portfolios of the *same* expected return, *any* inefficient benchmark induces infinitely many zero-beta portfolios at *all* expected returns. These make market risk premiums empirically unidentifiable and explain empirically found dead betas and negative market risk premiums. We characterize other misspecifications that arise when using inefficient benchmarks with models that require efficient ones. We provide a space geometry description and analysis of the specifications and misspecifications. We enhance Roll (1980), Roll and Ross's (1994), and Kandel and Stambaugh's (1995) results by offering a "Two Fund Theorem," and by showing the existence of strict theoretical "zero relations" everywhere inside the portfolio frontier.

JEL Codes: G10, G12
Key Words: Linear beta pricing, CAPM, expected returns, incomplete information, zero relation

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# LINEAR BETA PRICING WITH INEFFICIENT BENCHMARKS

## 1 Introduction

Current asset pricing models require mean-variance efficient benchmarks,[1] which are generally unavailable because of partial securitization and free float restrictions. We provide a pricing model that uses inefficient benchmarks, a two-beta model, with one induced by the benchmark and one adjusting for its inefficiency. While efficient benchmarks induce zero-beta portfolios of the *same* expected return, *any* inefficient benchmark induces infinitely many zero-beta portfolios at *all* expected returns. These make market risk premiums empirically unidentifiable and explain empirically found dead betas and negative market risk premiums. We characterize other misspecifications that arise when using inefficient benchmarks with models that require efficient ones. We provide a space geometry description and analysis of the specifications and misspecifications.

We enhance Roll and Ross's (1994) (RR) and Kandel and Stambaugh's (1995) (KS) results on "empirical" "zero relations" (benchmarks that have no explanatory power). We demonstrate the existence of "theoretical" *strict* zero relations, (not arbitrarily close to zero) *everywhere* inside the portfolio frontier (not just "close" to it). We complement Roll (1980, p.1020) "Three Fund Theorem" with a "Two Fund Theorem" regarding the spanning of the minimum variance frontier of the zero-beta portfolios with respect to some inefficient benchmark (Corollary 3, below).

Linear beta pricing motivated by equilibrium (as in the various CAPM versions), or arbitrage (as in the APT and various factors' pricing), stochastic discount factor pricing, or "risk neutral pricing," all identify the same pricing kernels. [See, for example, Hansen and Richard (1987), Huberman, Kandel and Stambaugh (1987), and

[1] We use the term *benchmark* to include *indices* and *portfolio proxies.*

Ferson (1995)]. Therefore, failure of a pricing method implies either failure of all methods or failure in implementing the method. The works of Roll (1977), RR, KS, and Jagannathan and Wang (1996) demonstrate failures in linear beta pricing with efficient benchmarks (LBPE) caused by internally inconsistent empirical implementations: the use of inefficient benchmarks with models for efficient ones. Hansen and Jagannathan (1997) assess specification errors in stochastic discount factor (SDF) models where SDFs are not perfectly correlated with efficient benchmarks. Thus, there is a sense in which the mean-variance representation in this paper is isomorphic to the SDF representation in Hansen and Jagannathan (1997).

With human capital, energy, art, and real-estate largely unsecuritized, and with free float restrictions, the likelihood is small to nil that a subset of the securitized free float, or other observable variables, spans efficient benchmarks in the space of market returns that are priced with respect to “what we care about.”

The above-cited works provide sufficient theoretical and empirical examples that imply that consistent correct pricing implementations require *inefficient benchmarks* to be used with *models for inefficient benchmarks*. Thus, we generally and simply provide, here, a linear beta pricing restriction for inefficient benchmarks (LBPI). The LBPI degenerates to LBPE when benchmarks become efficient, and characterizes critical misspecifications when inefficient benchmarks are used with models for efficient ones. While we give theoretical specifications for the errors resulting from the use of inefficient benchmarks with LBPE, we leave empirical applications of these specifications to future works.

Within a Markowitz world (a finite set of nonredundant risky securities with finite first two moments), we identify three sources of misspecification that arise while inconsistently using inefficient benchmarks with models for efficient ones: *i*) the omission of an addend in the pricing relation, *ii*) the use of incorrect risk premia/beta coefficients (due to the existence of infinitely many "zero-beta" portfolios at all expected returns), and *iii*) the use of unadjusted betas. We suggest the use of incomplete information equilibria to overcome unobservability of moments of returns. Our results are robust to the use of other pricing methods and regressions that produce positive explanatory beta power, including extensions such as multiperiod, multifactor, and the conditioning on time and various attributes.

Careful reading, of "Roll's Critique," Roll (1977), would have forewarned us of misspecification while using inefficient benchmarks with the traditional CAPM. Researchers have largely ignored this point, perhaps because of Roll's Critique and other seminal contributions.[2,3]

A Markowitz world with no further (equilibrium) assumptions induces exact linear relations between excess expected returns and betas (LBPE) with respect to any mean-variance frontier portfolio.[4,5] Quantitatively, this relation is identical to a classical CAPM relation. We say, and explain why below, that the LBPE is well defined for all reference portfolios excluding those with expected returns equal to that of the global minimum variance portfolio (GMVP).

In this context, we first develop a general and simple method to write the LBPI in terms of inefficient portfolios. We use the term "inefficient" portfolios to imply "non-frontier" portfolios. It is well known that LBPE can be written for any frontier portfolio.[6] Similarly, LBPI can be written by projecting on any frontier

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[2] Following the Merton (1972) mathematical development of the portfolio frontier, Roll (1977) first emphasized that all, and only, mean-variance frontier portfolios induce a CAPM; thus the only CAPM testable implication is the efficiency of the index. Roll also showed that the GMVP (global minimum variance portfolio) is an exception and that it induces beta equals to one on all assets.

[3] See discussion of notable exceptions towards the end of the introduction.

[4] See, for example, Feldman and Reisman (2003) for a simple construction.

[5] See discussion of the odd case of the global minimum variance portfolio in the appendix.

[6] This is, of course, the basis for Roll's Critique.

portfolio. Here, as in past related works [Hansen and Richard (1987), Hansen and Jagannathan (1997), for example] we choose to project on the efficient benchmark with which the inefficient one has the highest correlation. This also results in the simplest mathematical expression.

Our analysis emphasizes an essential implication: where the LBPE is well defined and where market portfolio benchmarks are inefficient, LBPE regressions are essentially misspecified because of three sources of misspecification. The first source of misspecification arises because the use of the LBPE with inefficient portfolio benchmarks inappropriately and incorrectly ignores a non-zero addend in the restriction. The second source of misspecification arises from the, above mentioned, existence of infinitely many "zero-beta" portfolios, and at all expected returns, *for any* inefficient benchmark. Thus, the identification of a correct "market risk premium," "excess return," or beta coefficient, is extremely unlikely. The third source of misspecification arises from the use of unadjusted betas, while adjusting the betas is required for inefficient benchmarks.

Where a LBPE is well defined, we very simply demonstrate that, as Roll (1980) showed, "Every nonefficient index possesses zero-beta portfolios at all levels of expected returns." [Roll (1980), p. 1011]. In particular, for any inefficient benchmark there is at least one and are practically infinitely many zero-beta portfolios of the same expected return. We show (see Section 2.4 below) that this implies that for any inefficient benchmark there is at least one portfolio and could be infinitely many portfolios that induce zero relations. We demonstrate a zero-relation case with both exogenously given and endogenously constructed zero relations. Consequently, a zero relation could be empirically detected.

Our work enhances RR results (see, for example, RR, Figure 1, p. 105) showing that the “region” that allows (empirical) zero relations might be “close” to the frontier. We are able to do so because we focus on a different cause for the zero relations. The cause that RR focus on is a zero "cross sectional covariance" between beta and expected returns as is the case of benchmarks corresponding to the GMVP, a case we call here “where the LBPE is not well defined.” The cause that we focus on is the zero coefficient on the beta caused by a zero-beta portfolio at the same expected return as the benchmark, inducing zero “risk premium” or “excess return” on all assets.

It enhances KS results because we demonstrate strict zero relations, not “almost zero relations” (see, for example, KS, page159, line 21).

The intuition behind this result and, more generally, behind the property that all (adverse) effects of using inefficient benchmarks in a model for efficient ones occur for all inefficient benchmarks, regardless of their “distance” from the frontier, is as follows. The frontier consists of endogenously constructed portfolios that have special properties not shared by non-frontier portfolios. Thus, moving away from the frontier even an infinitesimal distance discontinuously “switches regimes” in terms of these portfolios’ properties. If portfolios near the frontier had the same properties, they would also be solutions to the optimization problem that defines the frontier; they would have been part of the frontier, and we would have a “thick” frontier. Correspondingly, a change in the asset space naturally causes a shift of the frontier.

Where the LBPE is not well defined, we show that a theoretical zero relation between expected returns and betas (a zero coefficient of the betas in the LBPE restriction) may occur. It occurs, however, only under a degenerate indeterminate case that non-uniquely allows a theoretical zero relation as one possible relation out of infinitely many non-zero possible ones. This occurs where

- the reference portfolio is in a degenerate cone in the mean-variance space, at the line where expected returns are equal to those of the GMVP
- all securities have betas equal to 1 and the same expected return
- there is no zero-beta portfolio

The misspecifications that we point out above are robust with respect to the explanatory power of the betas. Also subject to the misspecification are LBPE regressions that use different procedures from Fama and French's (1992) and that produce positive beta explanatory power. The misspecification is also robust to various extensions, such as multiperiod, multifactor, and the conditioning on time and various attributes. This LBPI implication might be particularly beneficial as it is not clear that the RR KS, and Jagannathan and Wang (1996) essential implication—that LBPE regressions with inefficient benchmarks are meaningless—has been sufficiently internalized.

Because the real-world unobservability of moments of returns (a cause of the use of inefficient benchmarks) impairs the usefulness of the LBPI, we suggest implementing and testing incomplete information equilibria models developed to handle unobservable moments, as demonstrated in Feldman (2007), for example.

For a simple construction of the LBPI, we use an orthogonal decomposition similar to the one in Jagannathan's (1996) finite number of securities version of Hansen and Richard's (1987) conditioning information model.

Roll (1980) demonstrated that there is a (theoretical) zero relation between expected returns and betas for every inefficient portfolio where the LBPE is well defined. We provide intuition and simple construction of this result. Consider the hyperbola that an inefficient benchmark spans with the GMVP and, also, the (degenerate) hyperbola it spans with the frontier portfolio of the same expected return.

We demonstrate below that each of these hyperbolas includes a zero-beta portfolio to the inefficient benchmark and that these two zero-beta portfolios are of different expected returns.

Now, all infinitely many portfolios, expanding to all expected returns, on the hyperbola spanned by these two zero-beta portfolios are also zero beta with respect to the inefficient benchmark. We call such a hyperbola a “zero-beta hyperbola.” In addition, any zero-beta portfolio not on this hyperbola generates infinitely many additional portfolios that are zero beta with respect to the inefficient benchmark. There are vast regions where infinitely many such portfolios may exist numerically as well, as we demonstrate. Thus, we have at least one and possibly infinitely many zero-beta hyperbolas and on each such hyperbola infinitely many zero-beta portfolios.

Because any non-degenerate zero-beta hyperbola expands to all expected returns (as is the case for any non-degenerate hyperbola), it includes a portfolio with expected return equal to that of the inefficient benchmark. Moreover, there are infinitely many portfolios on each zero-beta hyperbola that induce incorrect “excess expected return” values (risk premiums/beta coefficients) in the LBPE. Thus, any inefficient benchmark induces incorrect pricing due to incorrect excess expected return premia with respect to infinitely many portfolios. When these excess expected returns are zero—that is, the expected returns of the zero-beta portfolios are equal to that of the inefficient benchmark—they induce zero relations. Of course, each of these infinitely many portfolios, whether inducing an incorrect excess expected return value or a zero relation, induces a pricing error.

While the analysis in this paper is done in a single period, its implications apply to multiperiod, multifactor models. This is because we can see the single period mean-variance model here as a “freeze frame” picture of a dynamic equilibrium where, because of the tradeoff between time and space, only the instantaneous mean and instantaneous variance of returns are relevant until the decision is revised in the next time instant.[7]

Roll (1977), RR, KS, and Jagannathan and Wang (1996) elaborately discuss the relation between expected returns and betas and its implications for regression estimates [see also the report of some of their results in Bodie, Kane, and Marcus (2005), Section 13.1, page 420]. Here, we introduce the LBPI and demonstrate properties of the relation; see RR, KS, and Jagannathan and Wang (1996) for detailed perspectives and references.

By providing a pricing specification for inefficient benchmarks, our results can be seen as closing a gap in the theoretical literature, which has provided a pricing specification for efficient benchmarks but not for inefficient ones. While the main practice of the empirical literature seems to be using inefficient benchmarks with models for efficient ones, substantial empirical literature has acknowledged and treated the issue of inefficient benchmarks. Notable exceptions to the main practice, in addition to Hansen and Jagannathan (1997) previously mentioned, include the following. Gibbons and Ferson (1985) develop tests under changing expected returns, unobservable market portfolio, or multiple state variables, implying changing risk premiums and conditioning information. Shanken (1987) develops a CAPM test for benchmarks that are sufficiently highly correlated with efficient ones. Gibbons, Ross and Shanken (1989) call the zero-beta portfolio that spans with an inefficient benchmark a given frontier portfolio the "active portfolio" and use it in statistical tests to determine efficiency. MacKinlay (1995) calls such a portfolio the "optimal orthogonal portfolio" and demonstrates that multifactor representation does not

[7] This is true for all preferences, myopic or non-myopic, diffusion state variables, and even dependent jumps. The instantaneous mean and variance will not be sufficient to span independent jumps.

overcome mispricing. More recently, Asgharian (2011) and Ferson (2012) use the "optimal orthogonal portfolios" under conditioning information. Lehmann (1989) develops cross-section efficiency tests, acknowledging an important property of inefficient benchmarks: the inducement of zero-beta portfolios at all expected returns. He proceeds to reject the efficiency of the benchmarks. In different contexts, Lee and Jen (1978) consider measurement errors effects, Green (1986) looks at consequences of inefficient benchmarks on deviations from the security market lines (SMLs), and Ferguson and Shockley (2003) examine the implications of omitting "debt" from the market portfolio and show that equity-only benchmarks induce understated betas. We, indeed, obtain a similar property for all inefficient benchmarks.

Section 2 demonstrates the results, Section 3 discusses implications, and Section 4 concludes. The appendix discusses cases where the LBPE is not well defined.

## 2 The LBPE Relation

Below, we introduce the model—a Markowitz world—and develop the analytical results.

### 2.1 Markowitz World and LBPE

In this section, we present the economy and write a LBPE using the following notational conventions: constants and variables are typed in italic (slanted) font, operators and functions in straight font, and vectors and matrices in boldface (dark) straight font.

In a market with $N$ risky securities, let $\mathbf{R}$ be an $N \times 1$ vector of rates of return of the securities, $R_i$, $i = 1,...,N$, and $N > 2$. We do not specify the probability distributions of the rates of return. Rather, we assume means and variances that are real finite numbers and a positive definite covariance matrix, $\mathbf{V}$, which implies that there are no redundant securities.[8] To avoid degeneracy, we assume that there are at least two securities with distinct expected returns and a non-frontier security. We call the vector of security expected returns **E**, the expectation operator $\mathrm{E}(\cdot)$, the covariance $\sigma_{ij}, \forall R_i, R_j$, the variance $\sigma_{ii} \triangleq \sigma_i^2, \forall R_i$, and the standard deviation $\sigma_i \triangleq +\sqrt{\sigma_i^2}, \forall i$.

Let some portfolio, say **a**, of the *N* market securities, be an $N \times 1$ vector of real numbers, with components $a_i$, $i = 1, \ldots, N$, where $a_i$ is the "weight" of security *i* in the portfolio and, unless otherwise noted, $\mathbf{1}^{\mathrm{T}}\mathbf{a} = 1$, where $\mathbf{1}$ is an $N \times 1$ vector of ones and the superscript T denotes the transpose operator. Let z be a zero-beta operator, i.e., z**a** is portfolio **a**'s zero-beta frontier portfolio; thus, by definition, $\sigma_{\mathrm{z}\mathbf{aa}} = \sigma_{\mathbf{a}\mathrm{z}\mathbf{a}} = 0$. We will call some portfolio that is uncorrelated with **a,** thus having a zero-beta with respect to **a**, *z***a**.[9] We call this world a Markowitz world.[10]

Let **q** be some frontier portfolio other than the GMVP. Portfolio **q** stands for a frontier index or reference portfolio. Then, we can write a Sharpe-Lintner-Mossin-Black (zero-beta) LBPE for **q**:

$$\mathbf{E} = \mathrm{E}(R_{\mathrm{z}\mathbf{q}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathrm{z}\mathbf{q}})]\frac{\mathbf{Vq}}{\sigma_{\mathbf{q}}^2}. \tag{1}$$

[11]

## 2.2 LBPI

In this section, we write a LBPI in terms of any portfolio, efficient or

[8] We define a redundant security as one whose return can be constructed by combining other securities.

[9] For simplicity of notation, and consistent with our notation convention, we use the operator z (z in straight font) on some portfolio **a**, z**a** to delineate the frontier portfolio that is uncorrelated with **a**; and we call some portfolio (not necessarily a frontier portfolio) that is uncorrelated with **a**, *z***a** [z is in slanted font (italics)]. The visual distinction between z**a** and *z***a** is subtle; but in context, there is little ambiguity and the introduction of additional notation is unnecessary. Also, as the GMVP induces no finite zero-beta portfolios, without loss of generality, we exclude z from operating on GMVP.

[10] Markowitz (1952), for example. See also Roy (1952).

[11] For a simple construction, see Feldman and Reisman (2003); for a geometric approach, see Bick (2004); and for a frontier expansion, see Ukhov (2005).

inefficient. The previous section's LBPE is, thus, a special case of this section's LBPI. We exclude, however, portfolios with expected returns equal to that of the GMVP, where the LBPE is not well defined, and discuss these in the appendix.

Let **p** be a portfolio with $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{q}})$ and $\sigma_{\mathbf{p}} > \sigma_{\mathbf{q}}$. Portfolio **p** stands for an inefficient benchmark that serves as a proxy to **q**. In a mean-standard deviation Cartesian coordinate system where the mean is on the vertical axis, **q** lies on the frontier and **p** lies inside the frontier to the right of **q**.[12,13]

We project $R_{\mathbf{p}}$ on $R_{\mathbf{q}}$, decomposing it into $R_{\mathbf{q}}$ and a residual return $R_{\mathbf{e}}$:

$$R_{\mathbf{p}} = R_{\mathbf{q}} + R_{\mathbf{e}}, \tag{2}$$

implying

$$\mathbf{p} = \mathbf{q} + \mathbf{e}, \tag{3}$$

where $\mathrm{E}(R_{\mathbf{e}}) = 0$, $\sigma_{\mathbf{qe}} = 0$, $\mathbf{1}^{\mathrm{T}}\mathbf{e} = 0$, $\sigma_{\mathbf{pq}} = \sigma_{\mathbf{q}}^2$,[14] $\sigma_{\mathbf{pe}} = \sigma_{\mathbf{e}}^2$, $\sigma_{\mathbf{e}} > 0$, and **e** is the weights vector of $R_{\mathbf{e}}$.[15] The orthogonal decomposition in Equations (2) and (3) is similar to those in Hansen and Richard (1987), (see Equation 3.7, p. 596), and Jagannathan (1996), (see Equation 1, p. 3).

We will now demonstrate why Equation (2) and the six following properties hold. Equation (2) and the first two properties hold because we can project any portfolio **p** on any portfolio **q** such that $R_{\mathbf{p}} = c + bR_{\mathbf{q}} + R_{\mathbf{e}}$, where *c* and *b* are constants,

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[12] **q** is the frontier portfolio with the highest correlation with **p**. See Kandel and Stambaugh (1987), Proposition 3, p. 68. This can be proven directly, spanning the frontier with **q** and z**q**, noting that **p** and z**q** are uncorrelated.

[13] For an examination of inefficient portfolios, see Diacogiannis (1999).

[14] In addition to the proof of this and the other properties below, the property of the mean-variance space that for any frontier portfolio **q** and any portfolio of the same expected return **p**, we have $\sigma_{\mathbf{pq}} = \sigma_{\mathbf{q}}^2$, which can be also proven by using the property that **q** is a GMVP with respect to **p** in the degenerated space of portfolios of the same expected return or by using the (elite) property that a frontier portfolio induces a beta of one on all portfolios of the same expected return.

[15] Note that from Equation (3), $\mathbf{e} = \mathbf{p} - \mathbf{q}$, thus, $\mathbf{e}$ exists and is unique.

$\mathrm{E}(R_{\mathbf{e}}) = 0$, and $\sigma_{\mathbf{qe}} = 0$. We achieve this if we choose $b = \frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}$, and $c = \mathrm{E}(R_{\mathbf{p}}) - \frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}\mathrm{E}(R_{\mathbf{q}})$.[16] The choice that $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{q}})$ implies that $c = 0$, $b = 1$, and, by left multiplying Equation (3) by $\mathbf{1}^{\mathrm{T}}$, that $\mathbf{1}^{\mathrm{T}}\mathbf{e} = 0$.[17] Equation (3) implies that $\sigma_{\mathbf{pq}} = \sigma_{(\mathbf{q+e})\mathbf{q}} = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{qe}}$ and that $\sigma_{\mathbf{pe}} = \sigma_{(\mathbf{q+e})\mathbf{e}} = \sigma_{\mathbf{e}}^2 + \sigma_{\mathbf{qe}}$. Together with $\sigma_{\mathbf{qe}} = 0$, we have $\sigma_{\mathbf{pq}} = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{qe}} = \sigma_{\mathbf{q}}^2$, and $\sigma_{\mathbf{pe}} = \sigma_{\mathbf{e}}^2 + \sigma_{\mathbf{qe}} = \sigma_{\mathbf{e}}^2$. Finally, because $\sigma_{\mathbf{qe}} = 0$, Equation (2) implies that $\sigma_{\mathbf{p}}^2 = \sigma_{\mathbf{q+e}}^2 = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{e}}^2 + 2\sigma_{\mathbf{qe}} = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{e}}^2$. Thus, the property $\sigma_{\mathbf{p}} > \sigma_{\mathbf{q}}$ implies that $\sigma_{\mathbf{e}} > 0$.

Equation (2)'s projection is similar to regressing $R_{\mathbf{p}}$ on $R_{\mathbf{q}}$. Equivalently, this is a market model presentation of $R_{\mathbf{p}}$, developed in Sharpe (1963).

Substituting $\mathbf{p} = \mathbf{q} + \mathbf{e}$ into Equation (1) yields

$$\mathbf{E} = \mathrm{E}(R_{\mathbf{zq}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathbf{zq}})]\frac{\mathbf{V}(\mathbf{p}-\mathbf{e})}{\sigma_{\mathbf{q}}^2}. \quad (4)$$

When we rearrange and define $\boldsymbol{\beta}_{\mathbf{p}} \triangleq \frac{\mathbf{Vp}}{\sigma_{\mathbf{p}}^2}$, and $\boldsymbol{\beta}_{\mathbf{e}} \triangleq -\frac{\mathbf{Ve}}{\sigma_{\mathbf{e}}^2}$ as vectors of market security betas with respect to portfolios **p** and **e,** respectively, Equation (4) becomes

$$\mathbf{E} = \mathrm{E}(R_{\mathbf{zq}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathbf{zq}})]\frac{\sigma_{\mathbf{p}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{p}} + [\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathbf{zq}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{e}}. \quad (5)$$

Equation (1) implies that portfolios with expected returns equal to that of z**q**

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16 The (orthogonal) decomposition $R_{\mathbf{p}} = c + bR_{\mathbf{q}} + R_{\mathbf{e}}$, $\sigma_{\mathbf{qe}} = 0$ implies $\sigma_{\mathbf{qe}} \triangleq \mathrm{COV}(R_{\mathbf{q}}, R_{\mathbf{e}}) = \mathrm{COV}(R_{\mathbf{q}}, R_{\mathbf{p}} - bR_{\mathbf{q}}) = \sigma_{\mathbf{pq}} - b\sigma_{\mathbf{q}}^2 = 0$, which, in turn, implies $b = \frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}$. If we choose *b* as implied and, in addition, choose *c* to equal $c = \mathrm{E}(R_{\mathbf{p}}) - b\,\mathrm{E}(R_{\mathbf{q}}) = \mathrm{E}(R_{\mathbf{p}}) - \frac{\sigma_{\mathbf{pq}}}{\sigma_{\mathbf{q}}^2}\mathrm{E}(R_{\mathbf{q}})$, we also have $\mathrm{E}(R_{\mathbf{e}}) = 0$ and accomplish the decomposition.

17 With $\mathrm{E}(R_{\mathbf{e}}) = 0$ and $\mathbf{1}^{\mathrm{T}}\mathbf{e} = \mathbf{0}$, **e** is an arbitrage portfolio.

are uncorrelated with $\mathbf{q}$.[18] In addition, $\mathbf{q}$ is $\mathbf{q}$. Thus, all portfolios with the same mean as $\mathbf{q}$ are uncorrelated with z$\mathbf{q}$. Therefore, because we have $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{q}})$, we also have $\mathrm{z}\mathbf{q} = z\mathbf{p}$. That is, the frontier portfolio that is zero beta with respect to $\mathbf{q}$ is zero beta with respect to all portfolios of the same expected return equal to that of $\mathbf{q}$, including $\mathbf{p}$ in particular. Thus, $\mathrm{E}(R_{z\mathbf{p}}) = \mathrm{E}(R_{\mathrm{z}\mathbf{q}})$, and we can rewrite Equation (5):

$$\mathbf{E} = \mathrm{E}(R_{z\mathbf{p}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\frac{\sigma_{\mathbf{p}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{p}} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}\boldsymbol{\beta}_{\mathbf{e}}\,. \qquad (6)$$

The intuition behind Equation (6) is straightforward. It is the LBPE where the efficient benchmark portfolio is written as the sum of two portfolios: one that is inefficient and one that is the difference between an efficient portfolio and the inefficient one. For parsimony and without loss of generality, the efficient and inefficient portfolios have the same expected return.[19]

Examining Equation (6), we identify three potential sources of misspecification that arise while using the LBPE with inefficient index portfolios. The first potential source of misspecification is, simply, ignoring the right-most addend of Equation (6).

The second potential source of misspecification is using incorrect excess expected return values due to the existence of portfolios that, although zero beta with respect to **p,** are of expected returns different from that of z$\mathbf{q}$. If, then, in empirical tests, the latter portfolios are used, the excess expected return values $[\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]$ are incorrect. We argue below that there are infinitely many such portfolios that could

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[18] Left multiplying Equation (1) by $\mathbf{a}^{\mathbf{T}}$ and rearranging yields $\sigma_{\mathbf{aq}} = \frac{\mathrm{E}(R_{\mathbf{a}}) - \mathrm{E}(R_{\mathrm{z}\mathbf{q}})}{\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathrm{z}\mathbf{q}})}\sigma_{\mathbf{q}}^2$, which demonstrates the property if $\mathrm{E}(R_{\mathbf{a}}) = \mathrm{E}(R_{\mathrm{z}\mathbf{q}})$.

[19] As will be emphasized later, Equation (6) is well specified only for some z**p**'s (out of infinitely many at all expected returns) and these are impossible to identify empirically.

cause this misspecification. In particular, when this excess expected return is zero, we say that the (inefficient) benchmarks induce zero relations. We examine these issues in the following sections.

The third potential source of misspecification is the use of unadjusted betas. Equation (6), the LBPI, adjusts the LBPE betas by multiplying them by the ratio of the inefficient benchmark's variance to the variance of a corresponding efficient benchmark of the same expected return. This ratio is greater than one, and it "becomes" one as the inefficient benchmark "becomes" efficient. As this misspecification holds for all inefficient benchmarks, it agrees with the results of Ferguson and Shockley (2003), who find that, omitting debt, equity-only (inefficient) benchmarks induce understated betas.[20]

We can rewrite Equation (6) such that it is additively separable in a traditional LBPE for **p** by writing the first addend without a beta adjustment. We can do so by recalling that $\sigma_{\mathbf{p}}^2 = \sigma_{\mathbf{q}}^2 + \sigma_{\mathbf{e}}^2$ (see above) and substituting for $\sigma_{\mathbf{p}}^2$ in Equation (6). We get

$$\mathbf{E} = \mathrm{E}(R_{z\mathbf{p}})\mathbf{1} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\boldsymbol{\beta}_{\mathbf{p}} + [\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{z\mathbf{p}})]\frac{\sigma_{\mathbf{e}}^2}{\sigma_{\mathbf{q}}^2}(\boldsymbol{\beta}_{\mathbf{p}} + \boldsymbol{\beta}_{\mathbf{e}}), \qquad (7)$$

where the first two addends on the right-hand side are a traditional LBPE with respect to **p**.[21,22]

## 2.3 LBPI, Space Geometry Graphical Representation

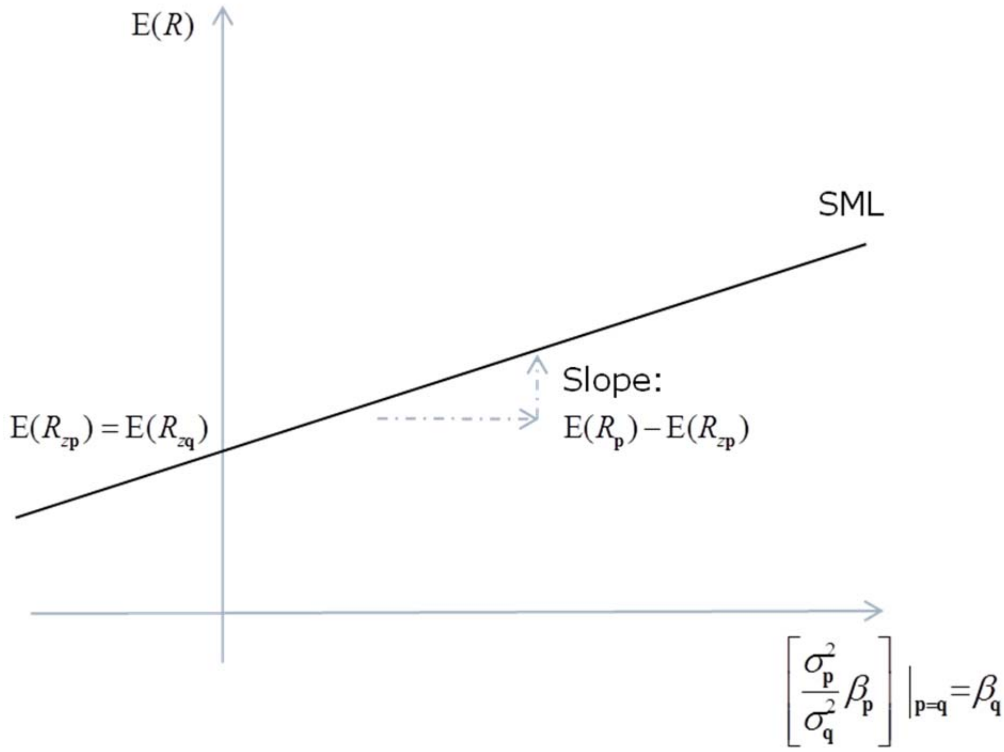
Diagram 1 below depicts the LBPE, described in Equation (1), as a special case of the LBPI. The LBPI degenerates to the LBPE where the inefficient benchmark

[20] Strictly speaking, the understatement is in the absolute value of the betas. Thus, for positive betas there is an understatement of the values, and for negative ones there is an overstatement.

[21] We thank Richard Roll for suggesting this presentation.

[22] Recall that Equations (6) and (7) are correct only for $z\mathbf{p}$ with $\mathrm{E}(R_{z\mathbf{p}}) = \mathrm{E}(R_{z\mathbf{q}})$.

**Diagram 1: LBPE as a Special Case of the LBPI**




becomes efficient; that is, where **p**=**q**, which guarantees that $[\mathrm{E}(R_{\mathbf{p}})=\mathrm{E}(R_{\mathbf{q}})]$, $[\mathrm{E}(R_{z\mathbf{p}})=\mathrm{E}(R_{z\mathbf{q}})]$, $\beta_{\mathbf{p}}=\beta_{\mathbf{q}}$, and $\sigma_{\mathbf{p}}^2/\sigma_{\mathbf{q}}^2=1$.[23]

Diagram 2 below is a graphical manifestation[24] of the LBPI, described in Equation (6). It is three dimensional: expected returns are on the vertical axis, the inefficient benchmarks', **p**, adjusted betas are on the horizontal axis pointing to the right, and the "correcting portfolio's," **e**, adjusted betas are on the axis that lies on the "horizontal plane" pointing "inward." The SML is the line spanned by points [*f*, *h*], is on the plane spanned by the vertical axis and the horizontal right pointing axis, and goes through the correct zero-beta rate. Indeed, this is consistent with the fact that on this plane the value of the correcting portfolio **e** is zero, that is, **p** merges into **q**.

[23] To simplify notation, diagrams use a scalar notation of beta, $\beta$, and the index of specific assets is depressed.

[24] We thank Soohun Kim for suggesting and presenting such a diagram, and we thank Xin Xu for adapting and creating the diagrams here.

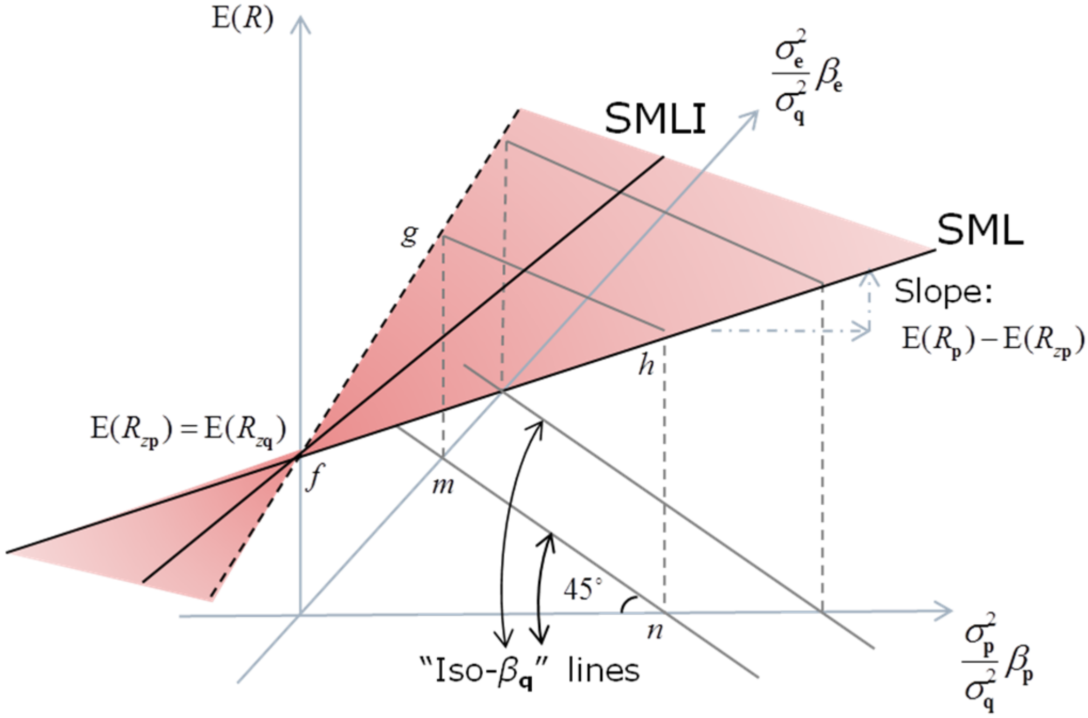
Diagram 1 describes this plane alone.

Only efficient benchmarks induce SMLs on this plane. We will call a SML that inefficient benchmarks induce SMLI. SMLIs lie on the colored hyperplane in Diagram 2 (the one on which [*f, g, h*] lie) and must have non-zero projections on the "correcting beta" axis. Thus, pricing with inefficient benchmarks only implies the use of incorrect SMLs/SMLIs. (This practice ignores a dimension in Diagram 2.) Of course, any efficient benchmark **q** induces one SML which corresponds to infinitely many SMLIs, generated by infinitely many inefficient benchmarks **p** and their corresponding corrections (to **q)**, the **e**'s.

All assets with certain $\beta_{\mathbf{q}}$ induce a line on the hyperplane on which [*f, g, h*] lie, for example, line [*g, h*]. This line is the locus of points, each on a different one of the infinitely many SMLIs. We call these lines and their projections on the horizontal plane "iso-$\beta_{\mathbf{q}}$" lines because all points on these lines correspond to the same $\beta_{\mathbf{q}}$ value. In Diagram 2, for example, the projection of line [*g, h*] is line [*m, n*].

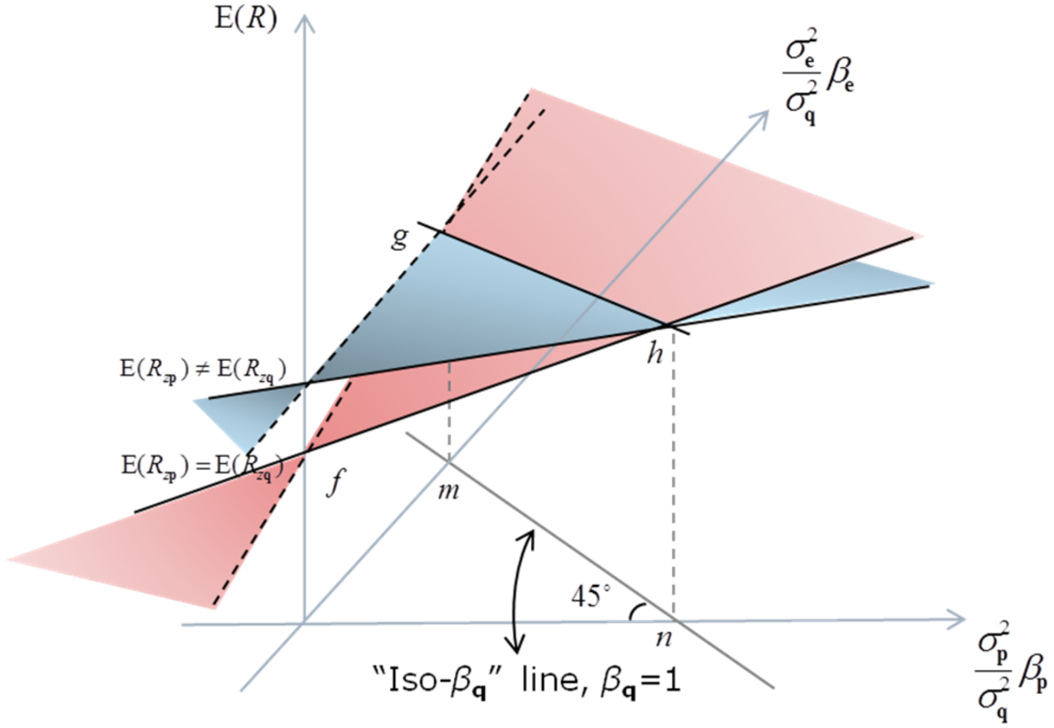
The restriction of same beta assets to only one line on the hyperplane is consistent with the property that projections on each of the horizontal axes are not independent. If axes had represented independent factors, projections would have been independent and could have spanned a hyperplane in the three-dimensional space. Here, however, once a value of (projection on) one horizontal axis is set, the value of (projection on) the other horizontal axis is determined. It must be the value that makes the sum of the values of both projections equal to the "true" beta value, $\beta_{\mathbf{q}}$. Thus, here, an asset spans a hyperplane of a hyperplane in the three-dimensional space.

**Diagram 2: LBPI (with Correct Zero-Beta Rate)**




If the zero-beta rate is the correct one and the only misspecification is the omission of the "correcting beta addend," SMLIs are on the colored hyperplane in Diagram 2. This hyperplane extends in all directions, not bounded by lines [*f, h*] and [*g, h*]. A correct SMLI, then, is a hyperplane of this hyperplane. If, in addition (because of the existence of infinitely many zero-beta portfolios of all expected returns for any inefficient benchmark), the zero-beta rate is incorrect, the intercept on the vertical axis is different, and the resulting hyperplane of induced SMLIs is a rotation of the hyperplane in Diagram 2. This rotation must be around the line [*g, h*], which is the "iso-$\beta_q$" line for $\beta_q = 1$, the only line on the hyperplane that is independent of the zero-beta rate. These are depicted in Diagram 3. The intersection of the resulting hyperplane with a correct SML/SMLI is reduced to one point only, which lies on the line [*g, h*].

**Diagram 3: LBPIs (with Correct and Incorrect Zero-Beta Rates)**




## 2.4 Where the LBPI is Well Defined

In this subsection, we provide a very simple construction of zero relations and a hyperbola of zero-beta portfolios for any inefficient benchmark where the LBPE is well defined.[25] See Roll (1980) for a comprehensive study of zero-beta portfolios' existence and properties. For exposition purposes, we start the discussion by showing the existence of both exogenous and endogenous zero relations.[26] Exogenous zero relations arise between the original assets in a Markowitz world. Note that in a Markowitz world, there is no restriction on the number of original assets that are uncorrelated. This number could be zero, two, or equal to the number of all original assets (diagonal covariance matrix.). Endogenous zero relations arise between assets or between portfolios that are not originally uncorrelated. Thus, an interpretation of

[25] For simplicity, we do not repeat the phrase, "where the CAPM is well defined" throughout the section.

[26] There is no loss of generality in the demonstration, however; and under cosmetic changes it is a proof.

this demonstration should be that in a Markowitz world there is no limit to the number of cases similar to those in the demonstration because of potential existence of exogenous zero relations.

***Demonstration.*** Assume a four-asset, **q**, **p**, **u**, and **v**, Markowitz world. If for $\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \\ \mathbf{u} \\ \mathbf{v} \end{pmatrix}$, we have $\mathbf{E} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then, solving for the portfolio frontier identifies **q** and **v** as frontier portfolios. Thus, we can view **p** as some inefficient benchmark and note that **u** (exogenously) induces a zero relation with respect to **p** as it is uncorrelated with it and has the same expected return. A more specific structure to support this demonstration could be as follows. Because **q**, **p**, and **u** have the same expected return, projecting **p** and **u** on **q** yields $\mathbf{p} = \mathbf{q} + \boldsymbol{\varepsilon}_{\mathbf{p}}$ and $\mathbf{u} = \mathbf{q} + \boldsymbol{\varepsilon}_{\mathbf{u}}$, respectively, where both $\boldsymbol{\varepsilon}_{\mathbf{p}}$ and $\boldsymbol{\varepsilon}_{\mathbf{u}}$ are of mean zero and uncorrelated with **q**. Then, setting $\sigma_{\boldsymbol{\varepsilon}_{\mathbf{p}}\boldsymbol{\varepsilon}_{\mathbf{u}}} = -\sigma^2_{\mathbf{q}}$ implies $\sigma_{\mathbf{pu}} = 0$. Thus, **u** is a zero-beta portfolio of and with the same expected return as **p**, inducing a zero relation. This could be the case, for example, where the **q**, **p**, **u**, and **v** are distributed according to a multivariate normal distribution. As **p** and **u** are original exogenously given assets, we call the zero relation that **u** induces with respect to **p** an exogenous one.

The intuition behind the existence of exogenous zero-relation portfolios as **p** and **u** in the demonstration above, and in general, is straightforward. It follows from the property that a Markowitz world specifies the first two moments of return distributions, leaving the freedom to further specify “distributions structure.” In order to keep "other things equal," a constraint on such "distribution structuring" is that it should not change the frontier.

We will now show that, within the demonstration's Markowitz world, there is an endogenously determined asset, a combination of **p** and **q**, where **p** and **q** are positively correlated, which induces a zero relation with respect to **p**. This asset, say z**p**, has a weight of 2 in **q** and -1 in **p**. Thus, the variance of z**p** and its covariance with **p** are

$$\sigma^2_{z\mathbf{p}} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = 2 \text{ and } \sigma_{\mathbf{p}z\mathbf{p}} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0 \,.$$

As the expected returns of **p** and z**p** are equal and they are uncorrelated, z**p** induces a zero relation with respect to **p**.

We will now show that the latter property is not coincidental to the last demonstration but, in fact, is a general property in this context: it exists for any inefficient benchmark in any Markowitz world. Consider some inefficient benchmark **p** and the frontier portfolio with the same expected return **q**. Consider now the (degenerate) hyperbola spanned by **q** and **p** only. We claim that on this single expected return hyperbola, **q** must be the GMVP. This is because **q** was already the GMVP for its expected return on a hyperbola that was spanned by **q**, **p** and additional assets. Removing the additional assets from the set of assets available to span the hyperbola could not have improved the optimum, that is, could not have allowed the creation of a portfolio with variance lower than that of **q**.[27] Thus, **q** must still be the GMVP on the hyperbola spanned by **q** and **p**.

It is a well-known property that a GMVP's covariance with all assets is equal

[27] The presence of additional assets is not necessary for the argument, of course. However, if there were no additional assets, **q** would have been the GMVP of the original hyperbola.

to a positive constant, its variance [see Huang and Litzenberger (1988), Section. 3.12, for example].[28] This property, together with the one that we demonstrated above—that **q** is the GMVP on the hyperbola spanned by **q** and **p**—implies that within any Markowitz world any inefficient benchmark **p** and the frontier portfolio of the same expected return **q** have a covariance matrix of the form $\begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix}$, $v_I > v_F > 0$, where $v_F$ is the variance of the frontier portfolio **q**, and $v_I$ is the variance of the inefficient portfolio of the same expected return **p**.[29] It, thus, becomes straightforward to identify a pair of weights, $(\alpha, 1-\alpha)$, of a portfolio that combines **q** and **p**, respectively, and form a portfolio that is uncorrelated with **p**. The weights of such a portfolio must solve the equation $\begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. Solving the equation, we get the well-defined solution $(\alpha, 1-\alpha) = \left( \frac{v_I}{v_I - v_F}, -\frac{v_F}{v_I - v_F} \right)$. Note that the weight of the frontier portfolio, $\frac{v_I}{v_I - v_F}$, is always positive and greater than one. It is the ratio of the variance of the inefficient portfolio over the variance increment of the inefficient portfolio over the frontier portfolio's variance. This ratio can be interpreted as related

[28] This property must also follow, and indeed follows, from a direct calculation of the covariance between the GMVP and any portfolio **a**. The (weight vector of the) GMVP [see, for example, Feldman and Reisman (2003)] is $\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}$. Thus, the covariance of the GMVP with any portfolio **a** is $\mathbf{a}^{\mathrm{T}}\mathbf{V}\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}$, a positive constant, independent of **a**. As **a** could stand for the GMVP, this covariance is also the variance of the GMVP.

[29] This property is also implied by the CAPMR. Rearrange the CAPMR for some portfolio **p** with respect to some non-GMVP frontier portfolio **q** as $\sigma_{\mathbf{q}}^2 \frac{\mathrm{E}(R_{\mathbf{p}}) - \mathrm{E}(R_{\mathrm{z}\mathbf{q}})}{\mathrm{E}(R_{\mathbf{q}}) - \mathrm{E}(R_{\mathrm{z}\mathbf{q}})} = \sigma_{\mathbf{pq}}$. Thus, for any portfolio **p** with the same expected return as **q**, this relation becomes $\sigma_{\mathbf{q}}^2 = \sigma_{\mathbf{pq}}$. In particular, for any **p** and **u** that have the same expected return as **q** and possibly $\sigma_{\mathbf{p}}^2 \neq \sigma_{\mathbf{u}}^2$, and applying the above relation twice, we have $\sigma_{\mathbf{q}}^2 = \sigma_{\mathbf{pq}} = \sigma_{\mathbf{uq}}$.

to a relative measure of inefficiency. We also note that the variance of the zero-relation portfolio is $\frac{v_I v_F}{v_I - v_F}$ $\left[\text{because, } \sigma^2_{z\mathbf{p}} = \begin{pmatrix} \frac{v_I}{v_I - v_F} \\ \frac{-v_F}{v_I - v_F} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} v_F & v_F \\ v_F & v_I \end{pmatrix} \begin{pmatrix} \frac{v_I}{v_I - v_F} \\ \frac{-v_F}{v_I - v_F} \end{pmatrix} = \frac{v_I v_F}{v_I - v_F}\right]$. We identify additional properties. Where $v_I = 2v_F$, as is the case in our demonstration above, the variance of the zero-relation portfolio is equal to the variance of the inefficient benchmark, that is, $\sigma^2_{\mathbf{p}} = \sigma^2_{z\mathbf{p}}$. (Of course, the expected returns of these portfolios are equal as well.) Further, $v_I > 2v_F$ $(v_I < 2v_F)$ implies $\sigma^2_{z\mathbf{p}} < 2v_F$ $(\sigma^2_{z\mathbf{p}} > 2v_F)$.

If we define a measure of relative inefficiency *RI*, $RI \triangleq \frac{v_F}{v_I - v_F}$, we can write the variance ratio of the zero-relation portfolio return over the frontier portfolio return as $\frac{\sigma^2_{z\mathbf{p}}}{v_F} = \frac{v_I}{v_I - v_F} = 1 + RI$. Then, we note that $\frac{\sigma^2_{z\mathbf{p}}}{v_F} = 1 + RI \xrightarrow[RI \to 0]{} 1$; that is, as the "inefficiency" of the benchmark grows, the zero-relation portfolio gets closer to the frontier. Conversely, $\frac{\sigma^2_{z\mathbf{p}}}{v_F} = 1 + RI \xrightarrow[RI \to \infty]{} \infty$; that is, the closer the benchmark gets to the frontier, the higher is the variance of zero-relation portfolio.

***Graphical representations of LBPI properties.*** We will now present eight graphs that manifest the LBPI properties. For easy exposition and without loss of generality, we follow the above demonstration. Figure 1 depicts the market's four assets **q**, **p**, **u**, and **v**, the portfolio frontier they induce, and the GMVP.

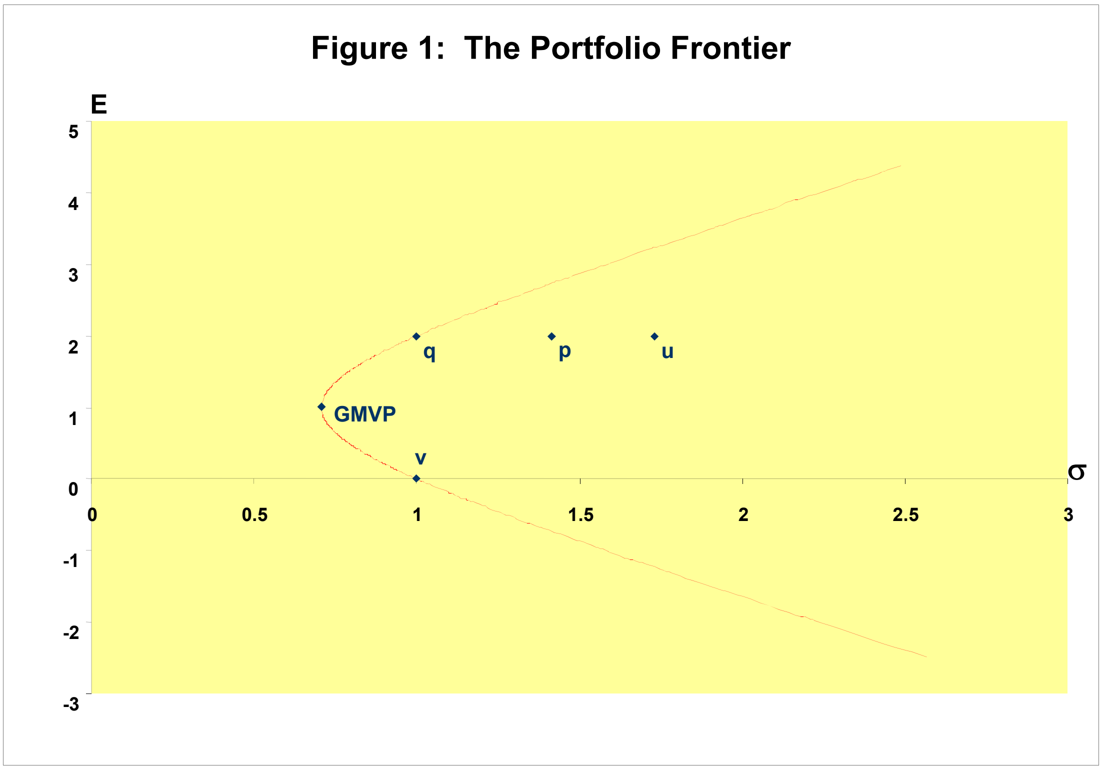



Figure 2 depicts the tangent to the efficient benchmark **q,** which is also the pricing line induced by **q**. Note that **v** is a zero-beta portfolio to **q** as it is at the level of the intercept of the tangent, or on the horizontal line.

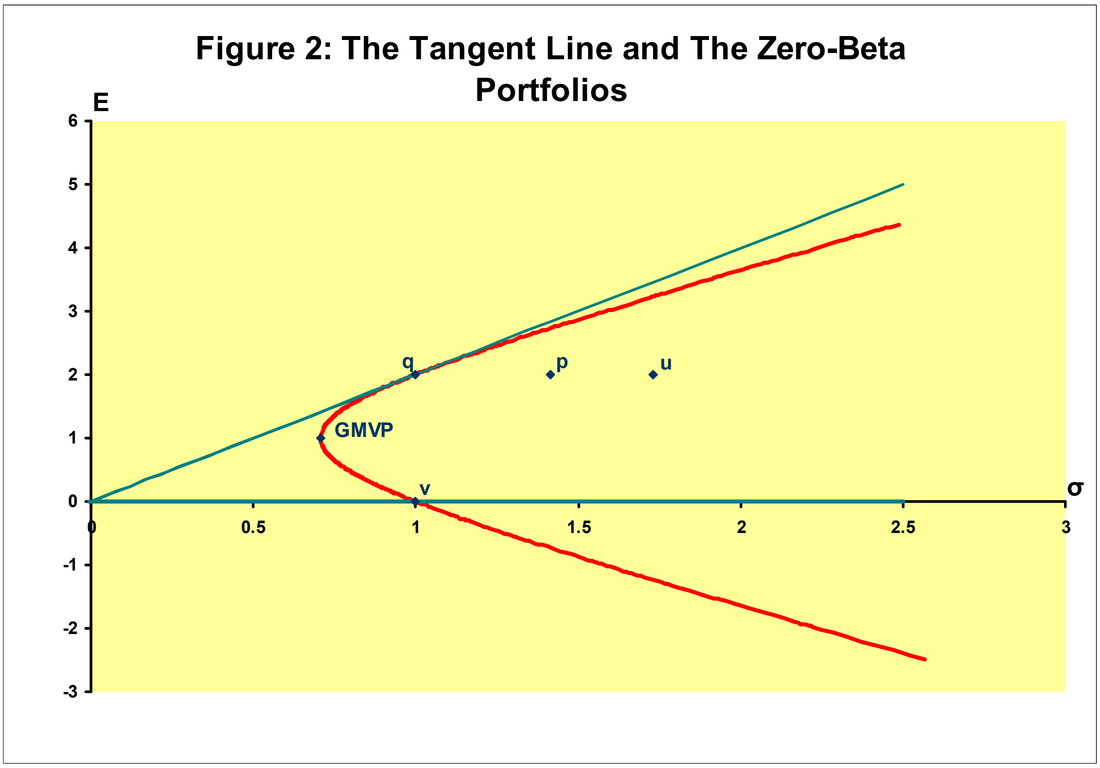


Figure 3 depicts the hyperbola spanned by **p** and GMVP. This hyperbola must have GMVP as its own GMVP.

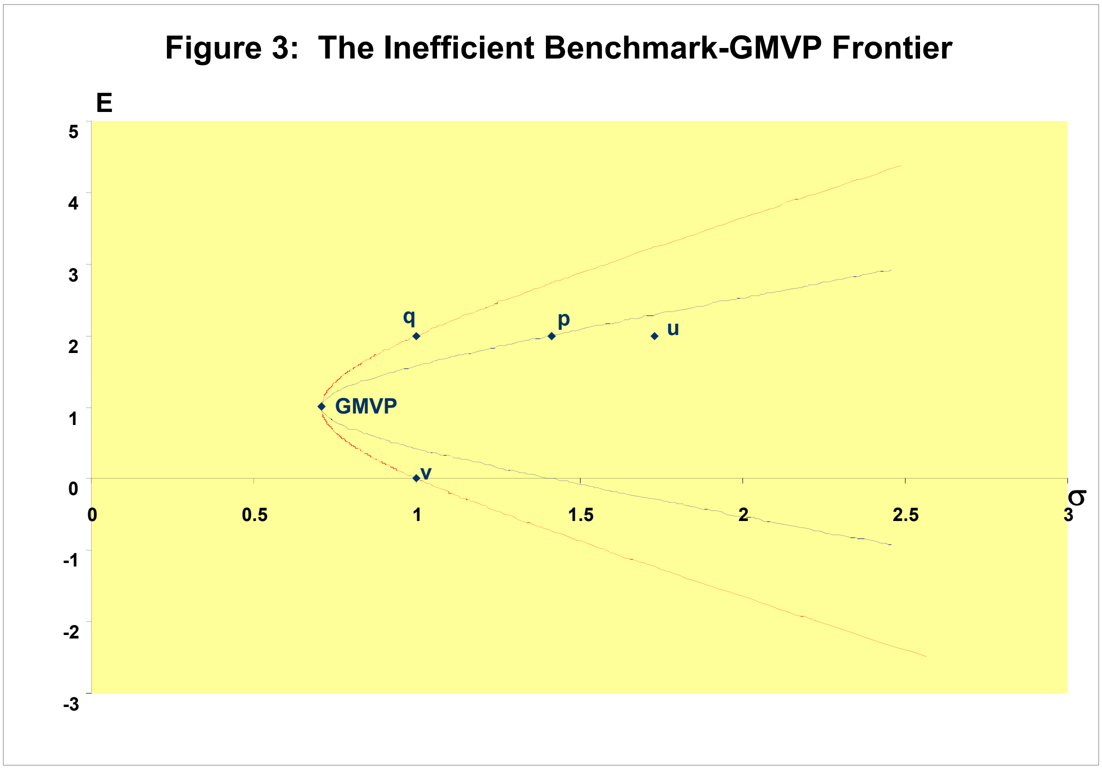
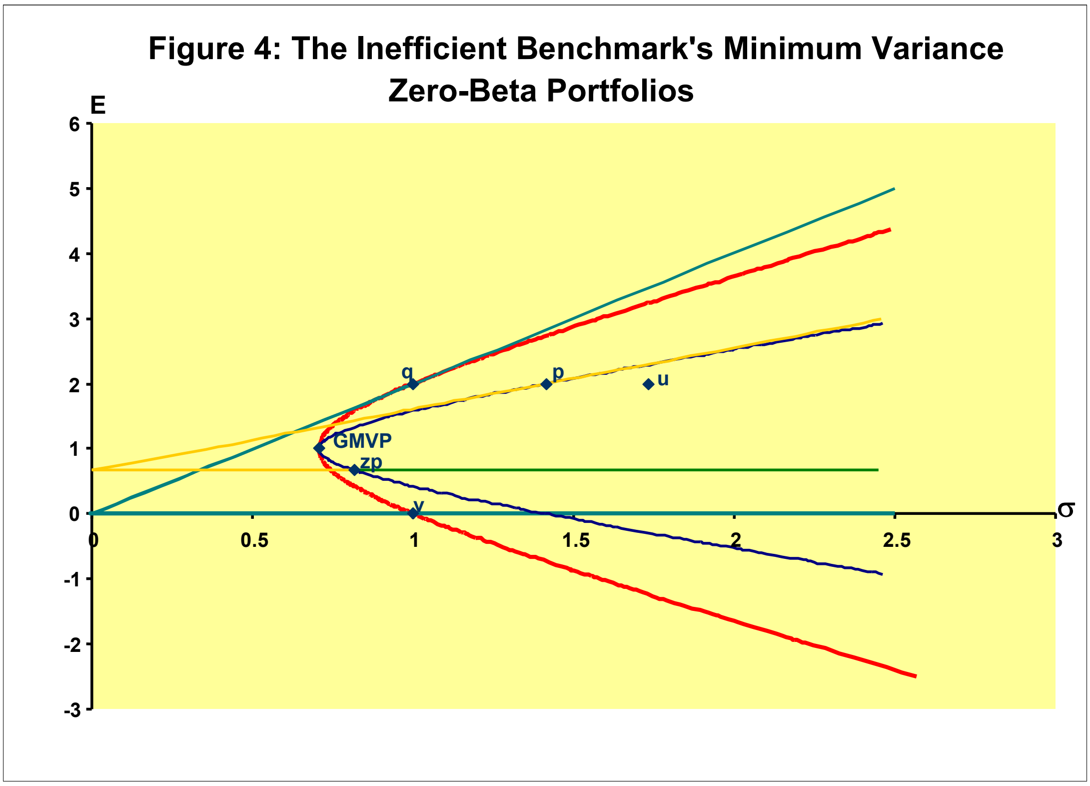



Figure 4 depicts the tangent to **p** on the hyperbola spanned by **p** and GMVP.



This tangent defines z**p**, **p**'s zero-beta portfolio on this hyperbola and a locus of higher variance zero-beta portfolios to **p** at the expected return of z**p,** on the green line.

**Figure 5: The Inefficient Benchmark's Endogenous Zero-Beta Frontier**

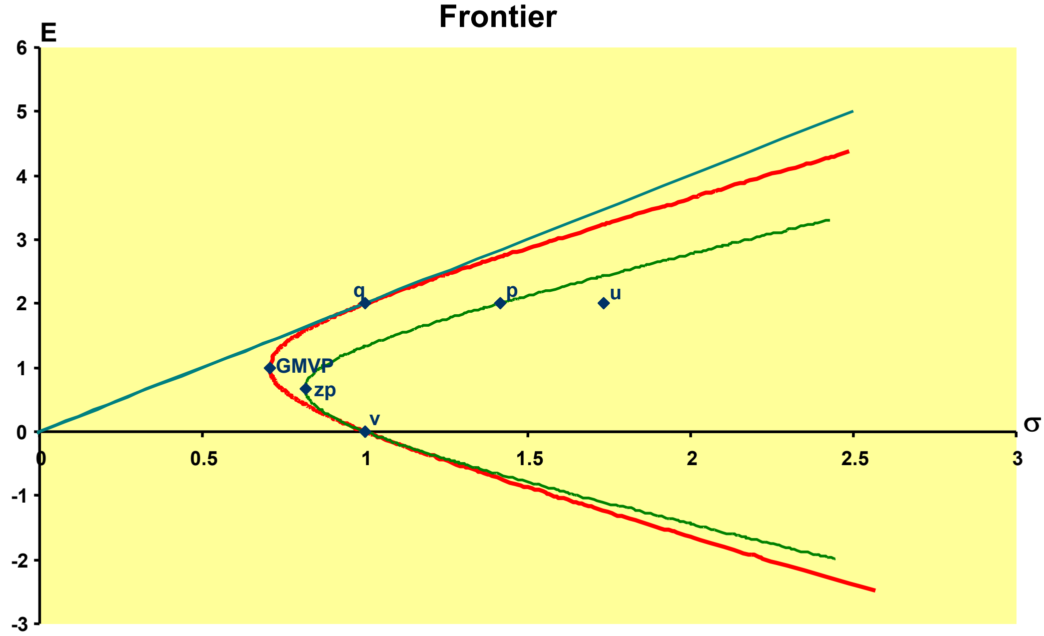
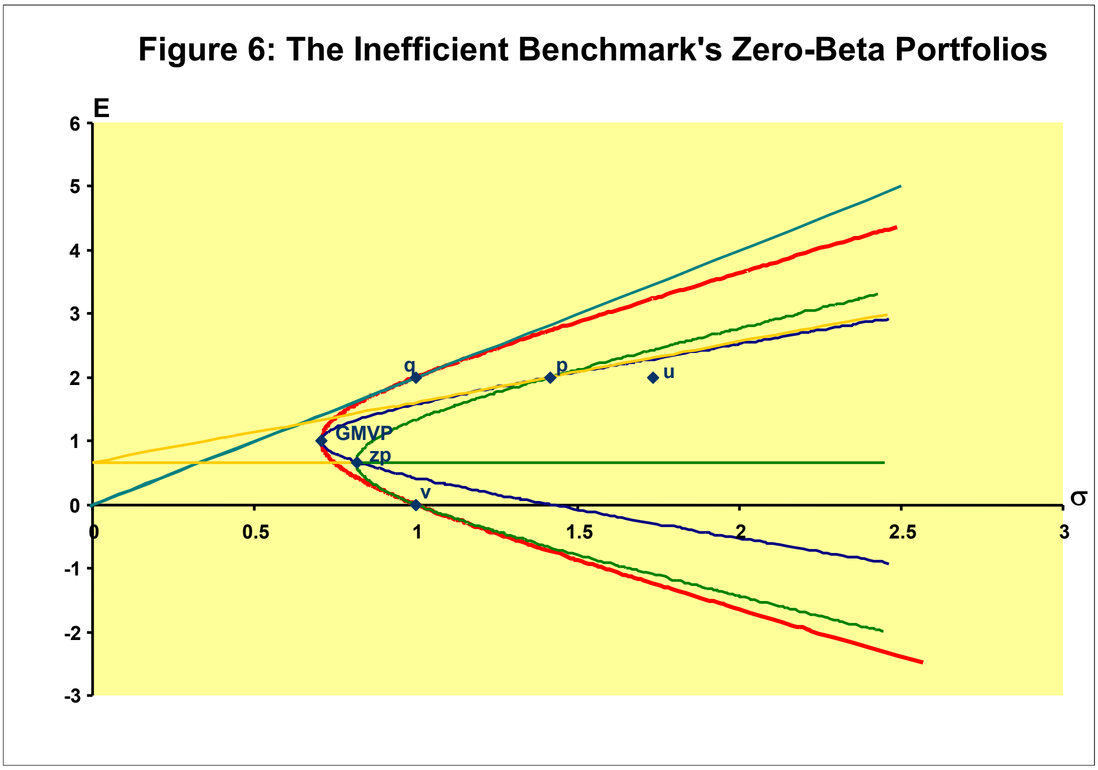



Figure 5 depicts the hyperbola spanned by **v** and z**p**. As both spanning portfolios are zero beta with respect to **p,** all this hyperbola's portfolios are also zero beta with respect to **p**. In our demonstration, this hyperbola goes through the expected value and standard deviation coordinates of **p**. As we demonstrate above, this is a special case that occurs when the variance of the inefficient benchmark **p** is double that of the corresponding efficient one **q**. The analysis above also demonstrates that if **p** "moves" to the left (right), the hyperbola moves to the right (left). Note that this frontier/hyperbola is the locus of the minimum variance zero-beta portfolios of **p**. Thus, for example, all exogenous zero-relation portfolios, induced by **u**, for example, will be contained within this hyperbola [see Roll (1980)].

Figure 6 superimposes Figure 5 on Figure 4 and depicts two loci of portfolios that are zero beta with respect to **p**: the horizontal line that passes through z**p** and the hyperbola spanned by **v** and z**p**. Combinations of portfolios from each locus further induce loci of portfolios that are zero-relation portfolios with respect to **p**.



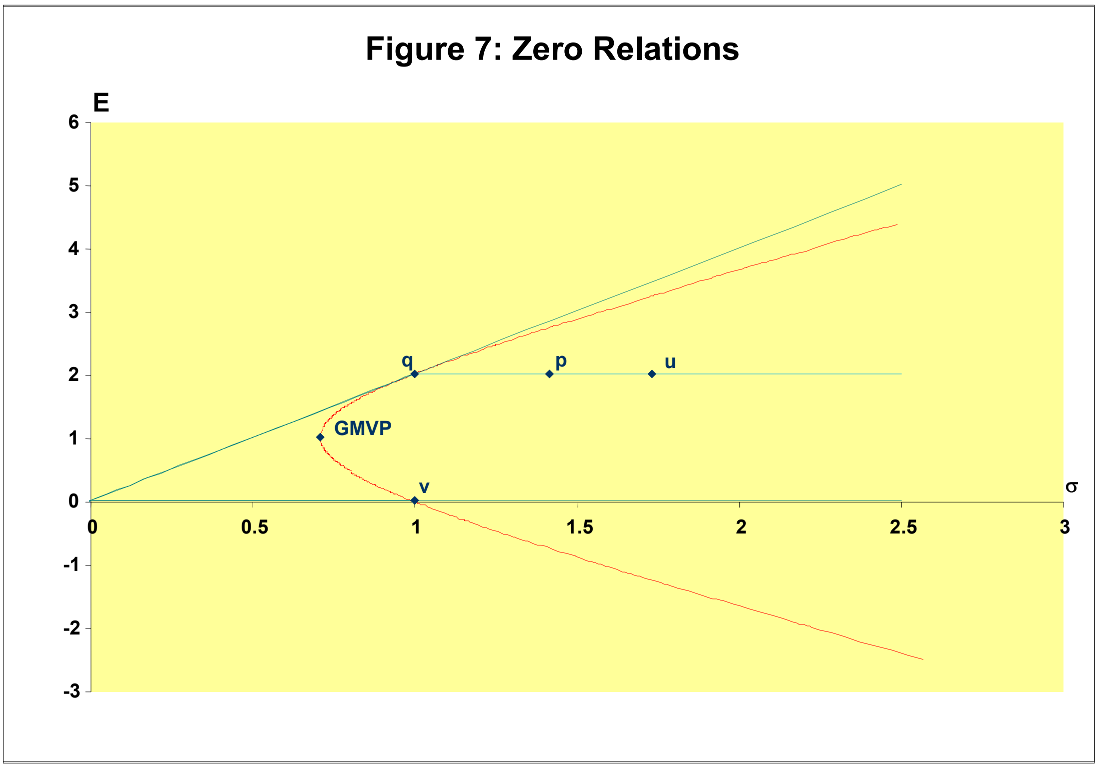
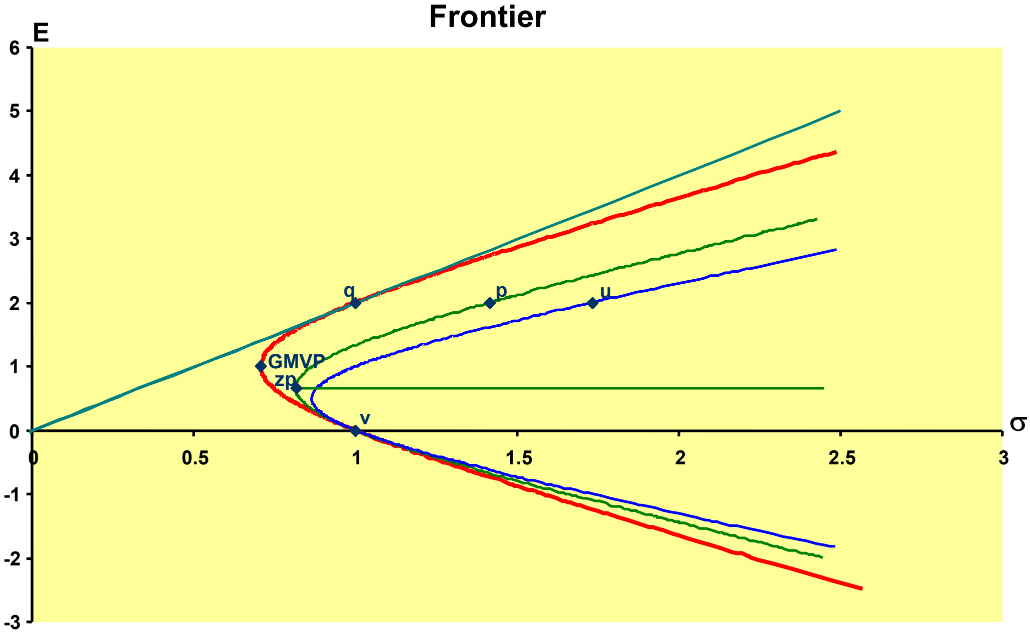



Figure 7 depicts the direct generation of a zero relation to **p** by combining **p** and **q**. As in the analysis above and in Figure 5, the zero-relation portfolio to **p**, in our demonstration, has the same expected value and standard deviation as **p**.

Figure 8 depicts an additional locus of portfolios that are zero beta with respect to **p**, generated by portfolio **u**, a market portfolio that is uncorrelated with **p**.

**Figure 8: The Inefficient Benchmark's Exogenous Zero-Beta Frontier**




We have, thus, proved and illustrated the following proposition and corollary.

***Proposition 1.*** *i) In a Markowitz world, any inefficient benchmark induces a zero relation. ii) Let, without loss of generality, the variance of some inefficient benchmark,* **p**, *be* $v_I$ *and that of the frontier portfolio of the same expected return,* **q**, *be* $v_F$, $v_I > v_F > 0$. *Then, the portfolio whose weights are* $\left(\frac{v_I}{v_I - v_F}, \frac{-v_F}{v_I - v_F}\right)$ *in* $(\mathbf{q}, \mathbf{p})$, *respectively, induces a zero relation with respect to* **p**, *and its variance is* $\frac{v_I v_F}{v_I - v_F}$.

***Corollary 1.*** *If the variance of the inefficient benchmark is double that of the frontier portfolio of the same expected return, then the zero-relation portfolio has the same* *variance (and, of course, the same expected return) as that of the inefficient benchmark. As the inefficient benchmark gets closer to the frontier, the variance of its zero relation grows to infinity. Conversely, as the variance of the inefficient benchmark grows to infinity, its zero-relation portfolio gets closer to the frontier.*

The following proposition identifies, for any inefficient benchmark, a zero-beta portfolio at a different expected return than that of the inefficient benchmark and its zero-relation portfolio identified in Proposition 1. It is the minimum variance inefficient benchmark's zero-beta portfolio among all the inefficient benchmark's zero-beta portfolios at all expected returns.

***Proposition 2****. [Roll (1980), Huang and Litzenberger (1988), Section 3.15]. Consider the hyperbola spanned by some inefficient benchmark and the GMVP. Then the GMVP is the GMVP of this hyperbola as well, and the zero-beta portfolio of the inefficient benchmark on this hyperbola is the minimum variance zero-beta portfolio of the inefficient benchmark, among all the zero-beta portfolios of the inefficient benchmark.*

The proof of the first part of Proposition 2 is similar to the proof of Proposition 1. The proof of the second part of Proposition 2, the identification of the inefficient benchmark's zero-beta portfolio as the minimum variance one among all its zero-beta portfolios, is demonstrated in Huang and Litzenberger (1988), Section 3.15, by Lagrange's method.

***Corollary 2.*** *The zero-beta portfolios, with respect to some inefficient benchmark, identified in Propositions 1 and 2, are of different expected returns.*

**Proof**. The zero-beta / zero-relation portfolio identified in Proposition 1 is of the same expected return as the inefficient benchmark. The zero-beta portfolio identified in Proposition 2 is on the other side, with respect to the inefficient benchmark, of the (non-degenerate) hyperbola spanned by the inefficient benchmark and the GMVP [see, for example, Huang and Litzenberger (1988), Section 3.15]. Thus, they must be of different expected returns.

As the two zero-beta portfolios identified in the propositions above are of different expected returns, they span a zero-beta hyperbola that extends to all expected returns. We state this property in the following proposition.

***Proposition 3***. *Any inefficient benchmark induces a hyperbola of zero-beta portfolios that extends to all expected returns. Such a hyperbola is the one spanned, for example, by the zero-relation portfolio identified in Proposition 1, and by the "minimum variance zero-beta portfolio" identified in Proposition 2. Moreover, this hyperbola consists of the minimum variance zero-beta portfolios at every expected return. The hyperbola includes one frontier portfolio, the (single) frontier portfolio that is uncorrelated with the frontier portfolio that has the same expected return as the inefficient benchmark.*

Roll (1980) attains the results of Proposition 3 in a different way. Using our approach, the proof of the first and second part of Proposition 3 is straightforward. Proving the latter part of the proposition, the property that the said hyperbola consists of the minimum variance zero-beta portfolios for each expected return, can be done by contradiction. Following the proof of Proposition 1, existence of a zero-beta portfolio with lower variance than that of the said hyperbola portfolio, will facilitate combining it with the frontier portfolio of the same expected return and constructing a portfolio with variance lower than that of the frontier portfolio. This is, of course, a contradiction.

Note, also, that any zero-beta hyperbola includes a single frontier portfolio. This frontier portfolio is the (only) frontier portfolio that is uncorrelated with the frontier portfolio of the same expected return as that of the inefficient benchmark that induces the zero-beta hyperbola. In fact, all portfolios of the same expected return are uncorrelated with a single frontier portfolio. On the other hand, all the portfolios uncorrelated with a frontier portfolio are of a single expected return. A consequence is that as an inefficient benchmark becomes efficient, the zero-beta hyperbola it induces degenerates/collapses to a degenerate (single expected return) hyperbola (or a line). See Roll (1980).

***Corollary 3.*** *Proposition 3 is a "Two Fund Theorem" for spanning the minimum variance frontier of the zero-beta portfolios with respect to some inefficient benchmark. The two portfolio identified in the proposition are a spanning basis, for example. The single zero-beta frontier portfolio could be used as well.*

The proof of the corollary is straightforward and omitted. This corollary complements the "Three Fund Theorem" of Roll (1980, p. 1020).

We reemphasize that although a zero relation generally induces a zero $R^2$ in a LBPE type regression, the choice of any zero-beta portfolio at any expected return—except the single expected return corresponding to the frontier zero-beta portfolio with respect to the benchmark (z**q** in our case)—induces a pricing error by inducing an incorrect excess expected return value / risk premium / coefficient on the beta in the LBPE. As we have demonstrated, there are infinitely many such portfolios for every expected return. The likelihood of identifying the "correct" zero-beta portfolio among the infinitely many seems to be negligible.

### 2.5 The LBPI Market Model: Correlated Explanatory Variables

We cannot say that the omitted addend in the LBPE is uncorrelated with or orthogonal to, the existing addends.

Following Sharpe (1963) and Black (1972), we can write the LBPI market model. To do that, we replace the explanatory random variable $R_{\mathbf{q}}$ in the LBPE market model with the difference in the random variables $R_{\mathbf{p}} - R_{\mathbf{e}}$.[30] Thus, we replace one market model addend, related to $R_{\mathbf{q}}$, with two addends related to $R_{\mathbf{p}}$ and $R_{\mathbf{e}}$ respectively. Recalling the construction method of the LBPI, it is easy to see that if **p** is indeed an inefficient portfolio (that is, if $R_{\mathbf{p}} \neq R_{\mathbf{q}}$), then $R_{\mathbf{p}}$ and $R_{\mathbf{e}}$ must be correlated. In other words, the two "new" addends in the LBPI market model must be correlated. This property might be material when considering the misspecification caused by ignoring, in implementations and tests, the addend related to $R_{\mathbf{e}}$. Thus, we cannot say that the omitted addend is orthogonal to, or uncorrelated with, the existing addends.

## 3 Implications

In this section we list a few implications of a Markowitz world.

### 3.1 Misspecification and Factor Pricing

Using models for efficient benchmarks with inefficient benchmarks data, as is the case of implementations and tests of the traditional CAPM, raises questions of internal consistency. Although one can nominally claim that these implementations and tests are under joint hypothesis,[31] these claims seem irrelevant under theoretical likelihood and ample empirical evidence of benchmarks' inefficiency (see also the discussion in Section 3.5 below).

Factor pricing models, however, do not suffer from internal inconsistency as they simply identify, endogenously or exogenously, factors that span a subspace of returns. If this subspace includes a portfolio that is on the portfolio frontier, there is

[30] Recall that by construction $R_{\mathbf{p}} = R_{\mathbf{q}} + R_{\mathbf{e}}$, thus $R_{\mathbf{q}} = R_{\mathbf{p}} - R_{\mathbf{e}}$.

[31] Model's validity and benchmark's efficiency

(with respect to this portfolio) a well-specified market pricing rule. The question then becomes how to empirically identify the single portfolio/relevant pricing kernel out of the infinitely many spanned by the factors, especially as previous works (RR and KS, for example) suggested that choosing the combination that induces the highest $R^2$ is not a relevant criterion.[32] If, however, all portfolios spanned by the factors are inefficient—that is, the frontier spanned by the factors is strictly inside the portfolio frontier and does not include even one portfolio that is tangent to the portfolio frontier—then the factor pricing will be subject to (all) the misspecifications that we identified above. These are true, of course, whether factors are endogenously identified or exogenously specified. Note that spanning two distinct frontier portfolios implies spanning the full return space. The likelihood is nil to negligible, we believe, that by using a subset of securities' free floating portions, we would be able to span even one portfolio of the frontier of "what we care about," which also reflects not fully securitized factors (human capital, energy resources, art, real-estate, etc.).

### 3.2 Misspecification of the LBPE and a Reemphasis of the Implication of Roll and Ross, Kandel and Stambaugh, and Jagannathan and Wang

Equation (6) is a well-specified LBPI and is distinctly different from the LBPE.[33] We say that when using inefficient benchmarks with the LBPE, we use a misspecified relation because we unjustifiably and incorrectly force an addend in the specified equation to be zero. This misspecification reemphasizes the important RR, KS, and Jagannathan and Wang (1996) results that demonstrate that it is meaningless to use inefficient benchmarks to implement regressions of CAPM, which is designed to use efficient benchmarks. For example, KS write in their abstract, "If the index portfolio is inefficient, then the coefficients and $R^2$ from an ordinary least squares

[32] Note that this portfolio is the only one whose zero-beta portfolios have the same expected return.

[33] As specified in Equation (1), for example.

regression of expected returns on betas can equal essentially any values….” Because real-world benchmarks are practically inefficient, such regressions based on the classical LBPE are misspecified. Jagannathan and Wang (1996, p. 41), provide an example of portfolios rearrangement, to which the LBPE should not be sensitive, that reduces the $R^2$ from 95% to zero.

The misspecification that we demonstrate is robust with respect to the explanatory power of the betas. Positive explanatory power of the betas does not imply that the well-specified LBPI would have resulted with the same values for $R^2$ and coefficients. In other words, CAPM regressions that unduly constrain a specification addend to be zero are subject to getting meaningless $R^2$ and coefficient values regardless of the $R^2$ and coefficients they produce. Thus, CAPM regressions that use different procedures from those used by Fama and French (1992) and that are able to produce positive beta explanatory power are also subject to the same misspecification. In addition, this misspecification is robust to multiperiod and multifactor models, and to those conditioning on various attributes.

A multitude of CAPM empirical studies followed the introduction of the CAPM in Sharpe (1964), Lintner (1965), Mossin (1966), and Black (1972), and the empirical works of Black, Jensen, and Scholes (1972) and Fama and Macbeth (1973). Curiously, however, the issue of the misspecification with respect to inefficient benchmarks, though highlighted by Roll's Critique [Roll (1977)] was largely ignored and was not attended to until the Fama and French (1992) results induced the declaration, "Beta is dead…". Please see references to notable exceptions at the end of the introduction section of this paper.

### 3.3 Infinitely Many Theoretical Zero Relations within a Markowitz World

While a main implication of this paper is the misspecification of the LBPE with inefficient portfolio benchmarks and the values of the misspecified coefficients and $R^2$ are immaterial, the prevalence and likelihood of zero relations has captured special interest in the literature. RR said, in their abstract, “For the special case of zero relation, a market portfolio proxy must lie inside the frontier, but it may be close to the frontier.” On page 104, they write, “Portfolios that produce a zero cross-sectional slope…lie on a parabola that is tangent to the efficient frontier at the global minimum variance point.” In addition, their Figure 1, page 105,[34] draws a boundary region that contains zero-relation benchmarks, one such portfolio being 22 basis points away from the portfolio frontier. We emphasize that where the LBPE is well defined, *any* inefficient benchmark (regardless of its “distance” from the efficient frontier) has at least one and possibly infinitely many portfolios that induce zero relations.

We say that for benchmarks whose expected returns are equal to that of the GMVP, the LBPI is not well defined because, as described above and in Appendix A, the GMVP has no zero-beta portfolios and the limit zero-beta rate is infinity. We identify, however, a degenerate indeterminate case that non-uniquely allows a theoretical zero relation: where all securities have the same expected return. The theoretical zero relation, however, is one possible relation out of infinitely many possible ones.

### 3.4 The Misspecification with Respect to Any Zero-Beta Portfolio

When considering the misspecification of the LBPE with inefficient benchmarks where the LBPE is well defined, it is important to note that zero-beta portfolios other than those noted below induce an incorrect excess expected return premium in the LBPE and, thus, a pricing error. This is in addition to the zero-beta portfolios with expected returns equal to that of the benchmark, which induce zero

[34] This figure is reproduced as Figure 13.1, in Bodie, Kane, and Marcus (2005), Chapter 13, page 421.

relations, and in addition to the zero-beta portfolios of expected return equal to that of the frontier zero-beta portfolio, which induce the correct excess expected return value in the LBPE. As stated above, there are infinitely many such portfolios for each expected return.

We can specify regions where the zero-beta portfolios could lie [see Roll (1980)], but considering the measure of these portfolios out of all portfolios might be irrelevant. Also, because a Markowitz world specifies only the first two moments of assets' return distributions, each point in the mean-variance space might represent more than one asset. These zero-beta portfolios induce zero relations or incorrect excess expected return values, thus, pricing errors.

### 3.5 A Robust LBPI and Incomplete Information Equilibria

Expected returns and variances, and thus the portfolio frontier, are unobservable. Moreover, assets that are correlated with returns on optimally invested wealth or consumption growth—human capital, real-estate, art, and energy, for example—are not fully securitized and traded. Thus, in all likelihood, real-world portfolio benchmarks are inefficient. Though Equation (6) is a robust LBPI in the sense that it holds for all benchmarks whether efficient or inefficient, an interesting question might arise regarding the usefulness of this relation, as inefficient benchmarks are unobservable as well. The answer to this question is twofold. First, observable or unobservable, the LBPI had better be well specified. Particularly, the LBPI expresses any portfolio as a combination of an inefficient one and the difference between an efficient portfolio and the inefficient one. The LBPE constrains this difference to be zero, limiting portfolios to be efficient. This constraint, however, is not satisfied; thus, the CAPM, which is a constrained (special case) of the LBPI, is misspecified. Because the LBPE is misspecified with inefficient portfolio benchmarks, we should use the LBPI in implementation and testing.

Second, to resolve the problem of unobservable means and covariances, we suggest the use of an incomplete information methodology. There we would identify a LBPE in terms of endogenously determined moments. We would use Bayesian inference methods (filters) to form these moments, conditional on observations. These observations would include (noisy) functions of the sought-after moments, such as prices, outputs, and macroeconomic variables. Such equilibria in a multiperiod, multifactor context were developed by Dothan and Feldman (1986), Detemple (1986, 1991), Feldman (1989, 1992, 2002, 2003), Lundtofte (2006, 2007), Björk, Davis and Landén (2010), and many others. Feldman (2007) includes a review of incomplete information works and a discussion of issues related to these equilibria.

## 4 Conclusion

The Sharpe-Lintner-Mossin-Black classical CAPM type relation implies an exact non-zero relation between expected returns and betas of frontier portfolios other than the GMVP. Because neither expected returns nor betas are directly observable and because not all assets that covary with the return on optimally invested portfolios or consumption growth are fully securitized, it is highly likely that asset-pricing implementations and tests (linear beta pricing, factor pricing, stochastic discount factor pricing, risk neutral measure pricing) use, or correspond to, inefficient portfolios benchmarks. Roll and Ross (1994), Kandel and Stambaugh (1995), and Jagannathan and Wang (1996) demonstrate that inefficiency of benchmarks might render LBPE regression results meaningless. They offer their finding as the reason behind the empirical results of Fama and French (1992) and others, and they intensively examine the relation between expected returns and betas.

We introduce the LBPI specification for any (inefficient) portfolio benchmarks. We suggest that because we use inefficient benchmarks, we should use the LBPI in implementations and tests and not use the LBPE, which is misspecified for use with inefficient portfolios. Three sources of misspecification arise when using the LBPE with inefficient index portfolios. One source of misspecification stems from ignoring an addend in the LBPI. The second source arises because of the infinitely many zero-beta portfolios, at all expected returns, which are likely to induce incorrect excess expected return values in the LBPE. And the third source of misspecification arises because betas of inefficient benchmarks are different from those of efficient ones.

Using the LBPE with inefficient benchmarks is a misspecification that renders the resulting coefficients and $R^2$ meaningless. This reemphasizes the RR and KS implication that the LBPE is misspecified for use with inefficient benchmarks, rendering CAPM regressions with inefficient benchmarks meaningless. We enhance the RR and KS findings on empirical zero relations by showing that theoretical strict zero relations exist everywhere inside the portfolio frontier. These misspecifications are robust to CAPM procedures that, unlike Fama and French (1992), find explanatory powers of betas and are robust to various extensions of the basic model, such as multiperiod, multifactor, and the conditioning on various attributes. To overcome the problem that means and covariances are not observable, we suggest implementing and testing incomplete information equilibria, as described in Feldman (2007), for example.

We also suggest the use of the LBPI explicit functional forms to increase the power of current procedures.

While the analysis in this paper is performed in a single-period mean-variance framework, its implications apply to multiperiod, multifactor models. This is because we can see the single-period mean-variance model here as a “freeze frame” picture of a dynamic equilibrium where, because of the tradeoff between time and space, only the instantaneous mean and instantaneous variance of returns are relevant until the decisions revision in the next time instant.

## Appendix A

### A1 Where the LBPI is Not Well Defined

In this appendix, we explore the case where the CAPMI is not well defined. This is the case where the benchmark is of the same expected return as the GMVP. Though it makes no sense to willingly choose a benchmark of the GMVP expected return, it is important to study this case because it is an empirical possibility, as the placements of the benchmark, GMVP, and the other assets/portfolios in a Markowitz world (the mean-variance space) are unobservable. Within this case, we further identify a special case, one where all securities have the same expected return.

Equation (6) implies that there is a zero coefficient of $\boldsymbol{\beta}_{\mathbf{p}}$ if and only if $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{z\mathbf{p}})$. The latter never happens with frontier zero-beta portfolios because if $\mathrm{E}(R_{\mathbf{p}}) > \mathrm{E}(R_{\mathbf{GMVP}})$ $[\mathrm{E}(R_{\mathbf{p}}) < \mathrm{E}(R_{\mathbf{GMVP}})]$, then $\mathrm{E}(R_{z\mathbf{p}}) < \mathrm{E}(R_{\mathbf{GMVP}})$ $[\mathrm{E}(R_{z\mathbf{p}}) > \mathrm{E}(R_{\mathbf{GMVP}})]$ (where z**p** is a frontier portfolio). See, for example, Huang and Litzenberger (1988), Equation (3.14.2), which follows Merton (1972).[35] Also, geometrically, $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{z\mathbf{p}})$ (where z**p** is a frontier portfolio) requires a flat frontier tangent (parallel to the standard deviation axis), a situation that cannot happen.[36]

We will now examine the case where $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{GMVP}})$. Because the

[35] Geometrically, this means that the above (below) GMVP frontier portfolios’ tangent intersects the expected return axis below (above) the GMVP expected return.
[36] See the discussion of the case where all securities have the same expected return (below).

covariance of the GMVP with any security equals the variance of the GMVP,[37] it induces a beta of one for all securities**;** there is no zero-beta portfolio, GMVP, and thus no zero-beta rate; and we say that the LBPE is not well defined (with respect to the GMVP). We also note that as the reference frontier portfolio moves (along the frontier) toward the GMVP, the absolute value of the zero-beta rate tends to infinity. When at least two securities have different expected returns, the LBPE does not exist. Geometrically, in this case, $\mathrm{E}(R_{\mathbf{p}}) = \mathrm{E}(R_{\mathbf{GMVP}})$ implies a frontier tangent having no intersection with (and parallel to) the expected return axis.

If, however, all market securities have the same expected return, the frontier consists of one point only, which is also the GMVP; and any benchmark has the same expected return as the GMVP. Thus, this is a special instance of the case described above where the LBPE is not well defined. Because all securities have the same expected return and the same beta, and because the zero-beta rate is not specified, there are infinitely many pairs of coefficients that average any constant (standing for the non-existent zero-beta rate) and one (standing for any security's beta) to equal securities' expected return. In particular, there is a pair of coefficients that allows a theoretical zero relation: if the constant that stands for the (non-existent) zero-beta rate is equal to securities' expected return, then a coefficient of one of the constants and a coefficient of zero of the betas explain all securities' expected returns. We call this a case of indeterminate degeneracy. We use the term *degeneracy* because expected returns degenerate to a single value, the hyperbola degenerates to a single point, the GMVP and the market portfolio degenerate to one portfolio, all betas degenerate to one, and a zero-beta portfolio and, thus, the zero-beta rate do not exist. We call this case *indeterminate* because there are infinitely many distinct pairs of

[37] See, for example, Roll (1997), also Huang and Litzenberger (1988), Section 3.12.

coefficients that explain expected returns, of which the theoretical zero relation is only one. Because of the latter property, we also say that the theoretical zero relation is non-unique.

**A2 Discontinuity and Disparity**

This section highlights properties related to the transition between the two cases, where the LBPE is and is not well defined. In a Markowitz world, there is an interesting "asymptotic discontinuity" when the reference portfolio becomes the GMVP. This discontinuity does not exist in a model with a risk-free asset. When there is a risk-free asset, the tangency portfolio becomes the GMVP as the risk-free rate goes to infinity (or negative infinity). Correspondingly, in analytical solutions, the weights of the frontier tangency portfolio go to the weights of the GMVP as the risk-free rate goes to infinity. Needless to say, the risk-free asset is always zero beta with respect to all risky portfolios, including the tangency portfolio.

In a zero-beta model, which is the model in this paper, as the tangency portfolio tends to become the GMVP, the zero-beta rate grows in absolute value and tends to infinity. However, as the tangency portfolio becomes the GMVP, its beta with any portfolio becomes one. There are no zero-beta portfolios, and thus no zero-beta rate.

Thus, in the "risk-free" case, zero-beta portfolios and a zero-beta rate (albeit possibly infinitely high) always exist, including the case where the tangency portfolio becomes the GMVP. In the zero beta case, in contrast, when the tangency portfolio becomes the GMVP, the beta it induces on all assets becomes one; there are no zero-beta portfolios and no zero-beta rate.

We call the phenomenon of "disappearance" of zero-beta assets and rate within the zero-beta model "asymptotic discontinuity" and the qualitative difference between the properties of the model with and without a risk-free rate "disparity."

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