Equilibrium Theory of Stock Market Crashes

Abstract

We consider an equilibrium in illiquid stock market where liquidity suppliers trade with investors and suffer significant trading losses. Such situation in the stock market was observed during a recent financial crisis. We find that the expected risk premium on the stock and its Sharpe ratio are positive and very large while the expected stock return volatility is a few times higher than in the liquid market. Investors sell stock shares due to their excessive leverage while market makers face significant trading losses that are partially compensated by the expected profits from buying the stock shares with very high expected Sharpe ratio. Moreover, the short–term predictability of stock returns in the illiquid market is much stronger than it is in the liquid market and can be positive or negative depending on the state of economy.

Keywords: General Equilibrium; Liquidity; Transaction Costs
JEL Classification: G11, G12

1 Introduction

It has been well documented that selling pressure from investors in the stock market rises during financial crises while the expected volatility of stock returns undergo a substantial increase.[1] Moreover, despite the realized decrease of the stock market price during financial crises, its *expected* return and the conditional Sharpe ratio rise dramatically at the time of crises.[2] The goal of our paper is to show that these stylized facts can be explained in a general equilibrium framework without assuming any irrational behavior of investors.

Our explanation is based on the assumption that liquidity providers suffers significant trading losses during financial distresses. To be general we assume that a liquidity provider could be a specialist, a dealer, a financial institution or an investor whom all we call market makers. A market maker trades with investors and supplies liquidity by either closing all transactions or by placing limit orders. Consequently, market makers face losses due to opportunity costs as well inventory and search costs. While these costs are not significant during normal times, they become dramatically aggravated during financial crises.[3] Meantime, investors post market orders and also face increased trading losses in terms of higher bid–ask spreads. Given the numerous stories on dramatic trading losses of market makers during financial turmoil, we assume that these losses are bigger than those of investors which we neglect for the purpose of tractability.[4] Facing significant trading losses makes market makers less willing to trade. On the other hand, investors, who face smaller losses on trading have strong incentives to trade. Therefore, the prices have to differ substantially from those in the liquid market so that market makers would trade more. We find that the risk premium, the expected Sharpe ratio and return volatility of the stock market behave the same way they are observed during financial crises.

We model illiquidity of the stock market by using convex transaction costs. These costs can be interpreted as losses due to the opportunity costs of not transacting, investment delay costs as well inventory and search costs. Convexity of the transaction costs is intuitive in illiquid market: As the size of transaction increases it becomes more difficult for a market maker to close a deal, therefore his losses per share on providing liquidity increases with the size of a trade. Moreover, because we treat the economy in continuous time, the convexity in the number of stock shares is converted into the convexity in the rate of share trading.

[1] For example, the CBOE S&P500 implied volatility index (VIX) which is used to measure short term expected volatility of the stock market returns increased by about three times during the financial crisis of 2008.

[2] See, for example, Kolb, Graham and Harvey (2011), Nagel (2012), and Martin (2013).

[3] The losses of market makers during a recent financial crisis were especially high in the OTC markets. Many market makers had to buy excessively many illiquid securities which they had hard time selling. Some dealers even withdraw from the markets. See, for example, “Markets: Exchange or Over-the-Counter” by Randall Dodd at www.imf.org.

[4] Even though the median quoted bid–ask spread of S&P 500 increased during financial crisis of 2007–2009, it did so by less than two times and the total quoted spread remained quite low. See, for example, Angel, Harris, and Spatt (2011). The quoted bid–ask spread in the OTC markets increases more significantly during the crisis but still was reasonably low. See, for example, “Got Liquidity?” at https://www.blackrock.com.

Consequently, a market maker endogenously chooses to trade at a finite rate and is not able to adjust his allocations quickly enough in response to idiosyncratic shocks. Note that convex costs have been used to model illiquidity by practitioners[5] and by academicians but to a smaller extent.[6]

We maintain the assumption that the stock market is competitive, implying that the change in price faced by a market maker or an investor due to his trading does not affect prices for the other market participants, and also does not depend on his previous trading. Importantly, we emphasize that, apart from explanation of the main features of the stock market observed during financial crises, our paper provides a general study of the effects that illiquidity of the stock market has on its returns and on the portfolio choices of economic agents in the presence of the asymmetric costs faced by agents in the market. As a benchmark case, we also consider an economy with a perfectly liquid stock market (with no transaction costs).

There are investors of the two types: Those who have logarithmic preferences and those who have exponential preferences. The exponential investors can sustain large negative shocks to the stock market price and allow logarithmic investors to avoid the default through the risk sharing. Most of the investors have logarithmic preferences while investors with exponential preferences are introduced for the technical reasons so that equilibrium will exist at all times. Overall, logarithmic investors comprise a marginal agent in the economy who defines the behavior of the stock prices in the most states of economy.

We find that an illiquid economy could be in states where investors have very long positions in the stock market which has very high expected returns and Sharpe ratio. Investors are willing to have an excessive leverage only if the stock market has large expected Sharpe ratio. In these states investors sell stock shares to market makers and buy liquid T-bills. On the other side of trades, market makers have small positions in the stock market. They are willing to be underexposed to securities with abnormally high expected Sharpe ratios, since buying them quickly is too costly for them. Note that investors do not become excessively leveraged in a liquid economy since they can immediately adjust their allocations by means of risk sharing with market makers.

If the aggregate investor is sufficiently wealthy and has significantly long or short positions in the stock, then the stock return volatility becomes very large. This happens because of the wealth effect on this aggregate investor. If his position is long (short) and becomes longer (shorter) then the stock price decreases (increases) causing the aggregate investor to loose money on his current position in the stock market. The losses make the aggregate investor sell stock shares when stock price is falling and buy them when the stock price is rising. Therefore, he further decreases the stock price when it falls or increases it when it rises causing the stock return volatility to increase. These results explain the increase in stock return volatility observed during the financial crises of 1929, 1987, 1997, and 2008 when the

[5]For example, see Grinold and Kahn (2000).

[6]See Heaton and Lukas (1996), Cetin and Rogers (2007), Isaenko (2010), Garleanu and Pedersen (2012), and Vayanos and Woolley (2012).

volatility jumped to at least twice the level it was before these distresses. Note that in a liquid market the wealth effect from the aggregate investor is offset by a substitution effect from market makers who makes the stock return volatility remain approximately the same across all states of the economy. The wealth effect from the aggregate investor becomes dominant in an illiquid market because trading by market makers is substantially limited by the presence of the transaction costs.

It follows that our model predicts the states of economy that are observed during financial crises: If investors happen to have excessively long positions in the stock market, then they aggressively sell their shares to market makers, the stock market is expected to be very volatile while the expected stock returns and Sharpe ratio significantly increase.

Besides states where the conditional Sharpe ratio and the expected returns are very high, our model also predicts states where the stock market has significantly negative conditional returns and the Sharpe ratio. The conditional returns on the stock market are negative when investors are short in this market. Such a situation cannot exist in a liquid economy where risk premium is positive in every state of the economy and an investor is always long in a stock market. It follows that the introduction of stock market illiquidity in the economy does not have to result in higher conditional premiums for holding the stock market. Market makers are willing to keep long positions in the stock market that is expected to fall due to the transaction costs that they have to pay to sell shares quickly. A possibility of a negative liquidity premium is also manifested in Vayanos (1998) and Longstaff (2009). These premiums are, however, rather small in magnitude to result in a negative risk premium.

For a better intuition on stock returns in illiquid market, we analyze the short–term predictability of stock market returns. We find that the autocorrelation coefficient of short–term stock returns in a liquid economy is negative and its magnitude does not exceed a small fraction of a percent. On the other hand, its value in an illiquid economy could be a few hundreds times bigger than it is in a liquid economy. It is negative for the most states and is highly significant where the wealth effect is strong. If the risk of the stock market allocation by the aggregate investor is excessive then the autocorrelation coefficient is positive and large. The coefficient is positive because the marginal stock premium with respect to the wealth of the aggregate investor is positive if he is very long in the stock market and negative if he is very short. At these states, a negative (positive) shock to the stock price is likely to follow by a further decrease (increase) of this price tomorrow. The logarithmic investors avoid default in these states by sharing risk with the other investors and market makers.

We provide a comparative statics by changing the calibration that affects the stock market liquidity. We find that decreasing the transaction cost results in a weakening of the wealth effect and, therefore, in a smaller deviation of the stock price from the one when the stock market is perfectly liquid (we call this deviation as mispricing). Conversely, a longer horizon for market illiquidity makes the mispricing more pronounced. Furthermore, a smaller percentage of investors in the economy makes the stock market less liquid resulting in a significantly stronger mispricing in all states of the economy. Finally, using the comparative statics and trading strategies of market makers, we discuss the possible microstructure and policy improvements that would mitigate the effects from financial crises.

Our study is related to recent papers by He and Krishnamurthy (2012, 2013). The authors model a financial crisis in a general equilibrium in which specialists trade risky assets for households and are financially constrained by contracts with them. In the presence of a negative shock to his wealth, a specialist becomes excessively leveraged and requires a significant increase in the expected stock returns and Sharpe ratio of risky assets. A leverage effect, however, does not result in a substantial increase in the volatility of asset returns which is observed empirically.[7] Our model complements that of He and Krishnamurthy. It suggests that besides the funding illiquidity faced by specialists, the behavior of the stock market observed during crises can also follow from trading liquidity faced by liquidity providers. Brunnermeier and Sannikov (2010) is another paper that studies an equilibrium in production economy with financial frictions and links intermediaries' financing position to risk premia.

Another important paper that is related to ours is that of Xiong (2001).[8] Xiong considers dynamic equilibrium in the presence of long–term investors, convergence traders, and noise traders where a risky investment is a spread between two mispriced securities. The author shows that convergence traders can increase the volatility of mispricing in states where their wealth is significant and the supply of the spread substantially fluctuates from its mean value. Xiong (2001) provides detailed economic intuition for the increased volatility of the stock mispricing as well as its conditional Sharpe ratio. This intuition is based on the wealth and substitution effects. Our paper complements that of Xiong (2001) in a few dimensions. First, we construct an equilibrium where all agents are rational. Second, we provide a new economic intuition on the nature of trading strategies and mispricing of stock returns. Finally, we avoid the critique of Loewenstein and Willard (2006) who showed that if the unlimited asset liability and a storage technology in perfectly elastic supply used by Xiong (2001) are replaced by a locally riskless bond market in which external supply of bonds is zero, then all the mispricing vanishes. The mispricing is present in our model even though the bond market clears at zero external supply.

Our paper contributes to the growing theoretical literature on a market's illiquidity. In addition to the above–cited references, important contributions include Lipman and McCall (1986), Amihud and Mendelson (1986), Grossman and Miller (1988), Grossman and Laroque (1990), Boudoukh and Whitelaw (1993), Heaton and Lucas (1996), Holmstrom and Tirole (2001), Longstaff (2001), O'Hara (2003), Eisfeldt (2004), Acharya and Pedersen (2005), Brunnermeier and Pedersen (2005), Duffie, Gârleanu, and Pedersen (2005, 2007), He and Mamaysky (2005), Jang, Koo, Liu, and Loewenstein (2005), Liu and Yong (2005), Cetin,

[7]He and Krishnamurthy (2012) show that the stock return volatility increases during crises. However, the biggest increase of the volatility that they show is less than 10% of its value in a normal state of the stock market. Our model combines both the leverage effect and the wealth effect. Both effects contribute to the dramatic increase of the risk premium and the Sharpe ratio while the wealth effect causes the volatility to spike.

[8]See also Kyle and Xiong (2001)

Jarrow, Protter, and Warachka (2006), Vayanos and Wang (2007), Vayanos and Weill (2008), Weill (2008), Brunnermeier and Pedersen (2009), Huang and Wang (2009), Longstaff (2009), Liu and Wang (2010), Pereira and Zhang (2010), Dai, Jin, and Liu (2011), Lagos, Rocheteau, and Weill (2011), Vayanos and Wang (2012), and many others.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 presents equilibrium in an economy with a liquid stock market. Section 4 analyzes an equilibrium in the economy with an illiquid stock market. Section 5 addresses the possible market adjustments aiming to mitigate financial crises. Section 6 concludes. Appendix A finds optimal strategies for the investors and presents the partial differential equations (PDE's) for the calculation of their indirect utilities, Appendix B derives the PDE for a value function of a market maker, Appendix C finds the PDE for the stock price in an illiquid market, Appendix D describes the numerical algorithm for finding an equilibrium, Appendix E outlines the solution of an equilibrium without transaction costs, while Appendix F provides details for finding the autocorrelation coefficient.

2 Economic Setting

We consider a markovian economy with a finite horizon T. We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, with uncertainty in the model being generated by a standard one–dimensional Brownian motion W_t which is also adapted.

2.1 Financial Securities

Similar to Grossman and Zhou (1996), we consider an economy where trading takes place continuously between time zero and time T. The traded securities are: Claims to a risky asset (a stock market) that is in positive net supply of 1, and a locally risk–free security (a bond), in zero supply, whose terminal payoff at T is always one unit of the consumption good. There is no consumption until time T, and there are no consumption goods until time T. The risky asset will not pay any dividend before time T, and at time T each unit of a risky asset will pay out a lump sum dividend at the amount of D_T (in units of consumption good). The lump sum dividend provides the consumption good at time T.

The view of D_T at time t is given by D_t following the process

$$dD_t = \mu_D dt + \sigma_D dW_t, \tag{1}$$

where μ_D and σ_D are constants and σ_D is positive. By definition, the price of the locally safe asset is always equal to one. The price of the risky asset is expressed in units of the price of the safe asset and follows

$$dS_t = \mu_{St} dt + \sigma_{St} dW_t, \tag{2}$$

where $\sigma_S > 0$. μ_S is a conditional dollar risk premium for holding the stock market per time–interval dt which we will simply call conditional risk premium. σ_S is a conditional standard deviation of a dollar return per time period dt which we will call conditional standard deviation of the stock return or stock return volatility.

2.2 Economic Agents

We assume that there are many individual economic agents of two types: Market makers and investors. An investor requires liquidity by placing market order. A market maker provides liquidity by either placing a limit order, finding another investor who is willing to close the transaction, or closing the transaction himself. A market maker could be a specialist, a dealer, an individual investor or a financial institution. Any individual economic agent behaves competitively.

If the stock market is illiquid then a market maker needs a significant time and effort to find a buyer or a seller of shares. Moreover, the more shares an investor is willing to trade within a given time interval Δt, the bigger efforts and costs per share a market maker has to face. Consequently, a market maker is not able to trade shares fast enough. As a compensation for the time and risks, a market maker earns a spread between prices paid and received by an investor. This spread also constitutes the cost of transaction faced by an investor. We assume that this spread is small and is nearly not enough to compensate a market maker for providing liquidity during a financial turmoil. Moreover, we neglect the trading losses of investors for the purpose of tractability.

Investors could be of the two types: Those who have logarithmic utility function and those who have exponential utility function. We assume CRRA preferences for the investors of the first type for the existence of the wealth effect in economy. The wealth effect makes investors sell (buy) the stock market shares when this market is falling (rising) and plays important role in a real economy. The second type of investors can absorb a negative shocks to the stock market price and therefore provide a channel for logarithmic investors to maintain their solvency through risk sharing. It will be shown that investors are marginal agents in economy so that the choice of preferences of market makers have a rather small influence on the equilibrium prices. Therefore, we assume an exponential utility function for a market maker allowing us to reduce the number of state variables in equilibrium. Furthermore, the proportion of logarithmic investors in the economy is denoted by λ_1, the proportion of exponential investors is λ_2, and the proportion of market makers is $1 - \lambda_1 - \lambda_2$. Because the role of exponential investors in the model is pure technical and allow the mathematical solution of the equilibrium to exists, we choose the weight of the exponential investors in economy to be very small.

For a better intuition on losses faced by a market maker let us first assume that trades take place at fixed discrete times $0, ..., t_k, t_{k+1}, ..., T$. If a market maker has to trade $|\Delta N_k|$ shares at time t_k, he will face liquidity losses per share that increase with the size of a trade and are given by $\hat{\alpha}_k |\Delta N_k|^{\varepsilon_k + 1}$, where $\hat{\alpha}_k$, and ε_k are positive numbers. Now let us assume that a market maker trades continuously in time at a finite rate of trading, so the number of shares that he purchases from time t_k to t_{k+1} is $\Delta N_k \approx u_k \Delta t_k$, where u_k is an average rate of trading at this time–interval and $\Delta t_k = t_{k+1} - t_k$. Therefore, the loss of a market maker from time t_k to t_{k+1} is $\alpha_k |u_k|^{1+\varepsilon_k} \Delta t_k$, where $\alpha_k = \hat{\alpha}_k (\Delta t_k)^{\varepsilon_k}$. Furthermore, we assume that α_k and ε_k are independent from Δt_k and from the state of economy k. The former assumption establishes the connection between continuous and discrete time frameworks.[9] It follows that a market maker pays the transaction costs $\alpha |u|^{1+\varepsilon} \Delta t$ within time–interval Δt. For simplicity of exposition, we also require that coefficient α be the same for purchasing and selling shares.

The convexity of the transaction costs does not allow the rate of his trading to be infinite. Naturally, share holding N becomes absolutely continuous and is given by

$$dN_t = u_t dt. \tag{3}$$

We conclude that market makers will trade slowly (at a finite rate). The convex transaction costs are appealing since they endogenize a finite rate of trading. In this regard, they are similar to the adjustment costs that are used in the literature on production economy to prevent firms from changing their capital stock too quickly.

2.3 Portfolio Choices by Investors

An investor of the first type maximizes his expected logarithmic utility function of the terminal wealth subject to the dynamic budget constraint:

$$\max_{N_1^i \in R} \quad E_0 \ln(X_{1T}^i) \tag{4}$$

$$dX_{1t}^i = N_{1t}^i \mu_{St} dt + N_{1t}^i \sigma_{St} dW_t, \tag{5}$$

$$X_{1t}^i = N_{1t}^i S_t + B_{1t}^i, \quad i = 1, ..., I_1, \tag{6}$$

where, X_1^i denotes an investor's wealth, N_{1t}^i is the number of shares that he holds, B_{1t}^i represents his investment in the locally risk–free security, and I_1 is a total number of investors of the first type in economy.

In Appendix A we describe how to find the indirect utility of a logarithmic investor and show that

$$N_1^i = X_1^i \frac{\mu_S}{\sigma_S^2}. \tag{7}$$

An investor of the second type maximizes his expected exponential utility function of the terminal wealth subject to the dynamic budget constraint:

$$\max_{N_2^l \in R} \quad E_0 \left(-\frac{1}{\gamma_2} \exp(-\gamma_2 X_{2T}^l) \right) \tag{8}$$

$$dX_{2t}^l = N_{2t}^l \mu_{St} dt + N_{2t}^l \sigma_{St} dW_t, \tag{9}$$

$$X_{2t}^l = N_{2t}^l S_t + B_{2t}^l, \quad l = 1, ..., I_2, \tag{10}$$

[9] Once the dynamic portfolio decisions by a market maker are found in the continuous time, one can easily adopt them to the discrete time trading by setting $\Delta N_k = u_k \Delta t_k$ and $\hat{\alpha}_k = \alpha / (\Delta t_k)^{\varepsilon}$.

where γ_2 is a coefficient of absolute risk aversion, X_2^l is the wealth of an investor of the second type, N_{2t}^l denotes the number of shares that he holds, B_{2t}^l represents his investment in the locally risk–free security, and I_2 is a total number of investors of the second type in economy.

Appendix A shows that in equilibrium $N_2^l = N_2$, where

$$N_2 = \frac{\mu_S}{\gamma_2 \sigma_S^2}\left(1 + \frac{X_1 g_{2X_1}}{g_2}\right), \tag{11}$$

and function $g_2(t, X_1, N_3)$ solves the PDE defined in Appendix A while variables X_1, N_3 stand for the wealth of an aggregate investor of the first type and share holdings of market makers to be introduced below.

2.4 Portfolio Choice by a Market Maker

A market maker maximizes the expected CARA utility function of his terminal wealth subject to his dynamic budget constraint:

$$\begin{aligned}
\max_{u_3^m \in R} \quad & E_0\left(-\frac{1}{\gamma_3}\exp(-\gamma_3 X_{3T}^m)\right) && (12)\\
dX_{3t}^m &= (N_{3t}^m \mu_{St} - \alpha |u_{3t}^m|^{1+\varepsilon})dt + N_{3t}^m \sigma_{St} dW_t, && (13)\\
dN_{3t}^m &= u_{3t}^m dt, && (14)\\
X_{3t}^m &= N_{3t}^m S_t + B_{3t}^m, \quad m = 1, ..., M, && (15)
\end{aligned}$$

where γ_3 is a coefficient of absolute risk aversion, u_3^m, X_3^m, N_3^m, and B_3^m stand for a market maker's rate of trading, wealth, number of stock shares and bonds, respectively, and M is a total number of market makers in economy.

The problem faced by a market maker can be solved only in a dynamic programming approach. For tractability, we assume that each market maker has the same initial allocation to the stock market. It will follow that each market maker trades at identical rate $u_{3t}^m = u_{3t}$ and has identical allocation to the stock market $N_{3t}^m = N_{3t}, \forall t \in [0, T]$. Therefore, our assumption allows us to remove N_3^m from the list of state variables. Because transaction costs payed by market makers dissipate from economy, the latter is incomplete and cannot be fully described by one state variable D. Instead, a state of economy is described by state variables D, X_1, N_3, and Y, where Y is the cumulative transaction payments paid by the aggregate market maker.[10] Indeed, state variables X_{1t}, N_{3t}, and Y_t depend on the historical realizations of state variable D_t and cannot be expressed as deterministic functions of t and D. However, because the conditional moments of the stock returns do not depend on Y

[10]We note that Y follows

$$dY = (1 - \lambda_1 - \lambda_2)\alpha |u_3|^{1+\varepsilon} dt,$$

where u_3 is the trading rate of an aggregate market maker, to be specified later. Moreover, the average wealth of exponential investors, X_2, and the average wealth of investors, X_3, are not state variables in economy since the controls of exponential investors and market makers are not affected by their wealths.

explicitly and the indirect utility function of a market maker allows separation of his wealth (see below), Y becomes redundant state variable in economy. For the same reason D is a redundant state variable for a market maker and investors.

We formulate the dynamic programming problem for a market maker in Appendix B. Solving this problem will provide us with his indirect utility function $V_3^m(t, X_3^m, X_1, N_3)$. We show in Appendix B that $V_3^m(t, X_3^m, X_1, N_3) = -\frac{1}{\gamma_3}\exp(-\gamma_3 X_3^m) f(t, X_1, N_3)$ and the trading rate of a market maker is given by

$$u_3 = -sgn(f_{N_3})\left(\frac{|f_{N_3}|}{\alpha(1+\varepsilon)\gamma_3 f}\right)^{\frac{1}{\varepsilon}}. \tag{16}$$

The presence of transaction costs forces a market maker to be in a state X_3^m, N_3^m in which his expected utility is not maximal for a given state of the economy X_1, N_3. Therefore, a market maker would like to change his allocations even in the absence of idiosyncratic shocks to the stock market. He trades in the direction of a state with a maximal utility, and switches the sign of his trading rate in this state. This behavior of market makers is a distinct feature of our model which is intuitive for an illiquid market (see also Longstaff, 2001) and is lacking in the existing general equilibrium models of an illiquid economy.

2.5 Equilibrium and its Calculation

Allocations and wealths of logarithmic investors do not have to be the same but have to satisfy relation (7). Let X_1 and N_1 be an average wealth and allocation across these investors. We capture the presence of investors of the first type by using an agent with wealth X_1 and allocation N_1 whom we call an aggregate investor 1.[11]

Definition 1 *An equilibrium is a price system* (μ_S, σ_S) *and a set of trading strategies* $(N_1^i,\ N_2,\ u_3)$, $i = 1, ..., I_1$ *of investors and market makers such that (i) individual agents choose their optimal portfolio strategies and (ii) the stock markets clears, that is* $\forall t \in [0, T]$

$$\lambda_1 N_{1t} + \lambda_2 N_{2t} + (1 - \lambda_1 - \lambda_2) N_{3t} = 1. \tag{17}$$

Note that the bond market clears as soon as the stock market does.[12] The dissipation of costs does not allow the existence of a representative agent in our economy. Nonetheless, the economy has a unique pricing kernel due to lack of arbitrage opportunities. The latter are absent because of investors who do not pay transaction costs. In Appendix C we show that the stock price, $S(t, D, X_1, N_3)$, is equal to $D + h(t, X_1, N_3)$, where h solves:

$$0 \;=\; h_t + \mu_D - \frac{\mu_S}{\sigma_S}\sigma_D + u_3 h_{N_3} + \frac{1}{2}\left(\frac{\mu_S}{\sigma_S}X_1\right)^2 h_{X_1X_1}, \quad h(T, X_1, N_3) = 0,$$

[11] We admit that the terminology "aggregate" is not precise here since an aggregate wealth of investors of the first type is $\lambda_1 X_1$ rather than X_1 .

[12] The bond market clearing condition is $\lambda_1 X_{1t} + \lambda_2 X_{2t} + (1 - \lambda_1 - \lambda_2) X_{3t} = S_t - Y_t$, where X_2, X_3 are the average wealths of market makers 2 and investors, respectively.

and

$$
\begin{aligned}
\mu_S &= \sigma_S^2 \Omega, && (18)\\
\sigma_S &= \frac{\sigma_D}{1-\Omega h_{X_1} X_1}, && (19)\\
\Omega &= \frac{1-(1-\lambda_1-\lambda_2)N_3}{\left[\lambda_1 X_1 + \frac{\lambda_2}{\gamma_2}\left(1+\frac{X_1 g_{2X_1}}{g_2}\right)\right]}. && (20)
\end{aligned}
$$

We will show that the stock market price in the economy with no transaction costs is very close to the dividend view D. Therefore, we call function h as a deviation of the stock price from fundamentals or simply the stock market's mispricing.

Note that the presence of only logarithmic investors and market makers in economy makes the former hold stock shares at time T. An unfavorable realization of the dividend payment may cause wealths of logarithmic investors become negative. Therefore, the presence of exponential investors is essential for the existence of equilibrium. The latter allow the logarithmic investors quickly liquidate their stock positions and avoid a default. The presence of two types of investors would be redundant if preferences of investors could support negative wealth and exhibit the desired wealth effect.

The equilibrium can be found only numerically by simultaneously solving the PDE's for functions g_2, f, and h. Taking into account that the weight of exponential investors in economy is very small allows us to find an equilibrium by simultaneously solving the PDE's for functions f and h and then for function g_2.[13] The numerical approach and the boundary conditions are described in Appendix D.

3 Equilibrium without Transaction Costs

Let us consider the economy described in the previous section but with no transaction costs. For the purpose of tractability we assume that the coefficients of risk aversion of investors of the second type and market makers are the same. Therefore, only two agents are present in the liquid economy: Agent 1 with logarithmic utility function and the weight in the economy of λ_1 and agent 2 with exponential utility function and the weight of $1-\lambda_1$. Appendix E presents the approach for finding an equilibrium in this economy. A state in this economy is completely defined by the current view of the dividend, D, and an assumed value of this view at some time in the past which is hidden in coefficient z_0.[14] Figure 1, Panel A shows the stock price (the thin dashed line), the conditional risk premium and the volatility of the stock return (the thick solid and dashed lines, respectively), the conditional Sharpe ratio

[13] It is assumed without a proof that the PDE's for f, h and g_2 have unique solutions and the numerical results converge to the true solutions in the limit of a very small time increment.

[14] To be more precise, z_0 is uniquely defined by setting D to some value D_0 for a given time in the past. We assume that this time occurred before time zero.

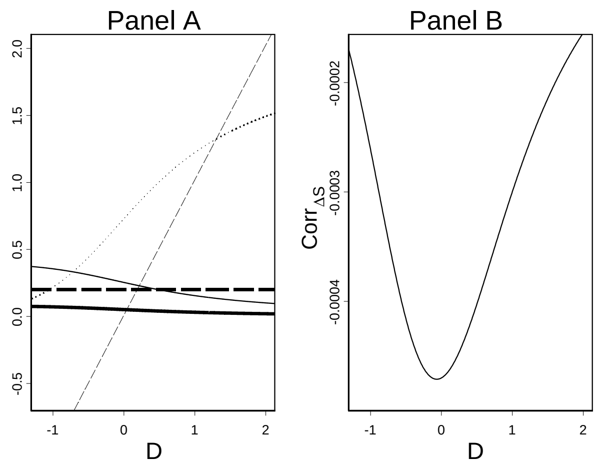

Figure 1: Panel A of this Figure shows the conditional risk premium, μ_S (the thick solid line), the volatility of the stock returns, σ_S (the thick dashed line), the stock price (the thin dashed line), the conditional Sharpe ratio (the thin solid line), and the stock allocation by agent 1, N_1 (the thin dotted line) as functions of the dividend rate, D. Panel B depicts the autocorrelation coefficient of stock returns, $Corr_{\Delta S}$, when $\Delta t = 0.02$. The stock market is liquid. We assume that $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $T = 0.5$, $\lambda_1 = 0.49, z_0 = 0.5$, and $t = 0$.

(the thin solid line) as well as the stock allocations by the investor, N_1 (the thin dotted line), versus D for the calibration to be used in the illiquid economy.[15] As seen from the panel, the stock price follows D very closely and the volatility of the stock return is very near to σ_D at any state of economy.

The existence of an equilibrium in this economy hinges on the presence of agent 2 who can sustain large negative shocks to his consumption. If D is very small and decreases then agent 1 reduces his stock holding, N_1, to zero while his wealth approaches zero as well. In these states, the stock market is dominated by agent 2, so that the conditional Sharpe ratio, the stock return volatility and the risk premium converge to $\gamma_2\sigma_D/(1-\lambda_1)$, σ_D and $\gamma_2\sigma_D^2/(1-\lambda_1)$, respectively (see Appendix E). The impact on the stock prices from agent 1 becomes significant in states where D is not too small. Since the coefficient of absolute risk aversion of this agent decreases with the growing stock price, his share of the stock market increases with D. In the limit of a very large D, the economy is dominated by agent 1 whose coefficient of absolute risk aversion becomes zero. He takes a very long position in the stock market and becomes much more wealthy than agent 2. Consequently, the conditional risk premium and the conditional Sharpe ratio converge to zero, the stock return volatility converges to σ_D and N_1 approaches $1/\lambda_1$.

Figure 1, Panel A shows that in any state of an economy both agents have long positions in the stock. Moreover, the stock allocation of agent 2 essentially follows the conditional

[15]We set $z_0 = 0.5$ for concreteness. We considered the liquid economy at various values of z_0 and confirm that all the conclusions of this section hold qualitatively at any feasible choice of z_0.

Sharpe ratio. Indeed, the stock allocation by investors with exponential utilities are not affected by their wealths and therefore are subject to only the substitution effect. On the other hand, agent 1 is also affected by the substitution effect which induces him to allocate less to the stock market as the dividend rate grows. Yet, he allocates more to the stock market with a higher dividend view since his wealth increases faster than the conditional Sharpe ratio decreases. Therefore, the wealth effect has a leading impact on the strategy of agent 1.

It follows that if the stock price experiences a positive shock then agent 1 buys more stock shares because of the wealth effect. This trading makes the volatility of the stock returns increase. In spite of this, the volatility barely changes due to the substitution effect from agent 2. Overall, the heterogeneity between agents preferences results in only small variations of the conditional volatility of the stock returns across different states of the economy. On the other hand, the conditional risk premium undergoes more significant variations. Yet, this premium always remains positive. Furthermore, our analysis shows that the risk premium and stock holdings of each agent remain positive in all states under various calibrations of economy including those with a negative return of the dividend view, μ_D. We conclude that the risk premium in a liquid economy cannot be negative at any calibration. Moreover, short initial allocations of market makers (investors) will not sustain in equilibrium and will be immediately adjusted to long ones by means of instant trading and price changes.

Since the stock holdings of agent 1 and agent 2 are $X_1\frac{\mu_S}{\sigma_S^2}$ and $\frac{\mu_S}{\gamma_2\sigma_S^2}$, respectively, the stock clearing condition suggests that $N_2 = \frac{1}{\lambda_1\gamma_2 X_1+1-\lambda_1}$. If the volatility of the stock holdings by agent 2, σ_{N_2t}, is defined from $dN_{2t} = \mu_{N_2t}dt + \sigma_{N_2t}dW_t$ then from Itô's lemma we find

$$\sigma_{N_2} = \frac{\gamma_2\lambda_1 X_1\mu_S/\sigma_S}{(\gamma_2\lambda_1 X_1 + 1 - \lambda_1)^2}.$$

We confirm that the stock holdings of both agents have infinitely large first–order variations. This conclusion is expected for the liquid market since an investor can trade an arbitrary large number of stock shares very quickly.

For the purpose of comparison with an illiquid economy we also find the instantaneous autocorrelation coefficient between stock returns:

$$Corr_{\Delta S}(t) \quad = \quad \frac{Cov_t(\Delta S_t, \Delta S_{t+\Delta t})}{\sqrt{Var_t(\Delta S_t)Var_t(\Delta S_{t+\Delta t})}} = \frac{S_{tD} + \mu_D S_{DD} + \frac{1}{2}S_{DDD}\,\sigma_D^2}{S_D}\,\Delta t, \qquad (21)$$

where $\Delta S_t = S(t+\Delta t) - S(t)$, Δt is very small, and the expectations are conditioned on the information at time t. Equation (21) is derived in Appendix F. Panel B of Figure 1 depicts the autocorrelation coefficient of the stock market returns for the time period of 0.02 years. The autocorrelation coefficient of the stock returns is negative and does not exceed 0.05% in magnitude. Negative and very small short–term predictability of stock returns follows from a negative and very weak sensitivity of the conditional risk premium to idiosyncratic shocks: A random increase in the stock price is likely to result in a slightly smaller risk premium causing a small decrease in the future stock price.

4 Equilibrium with Transaction Costs

In this section we show the results of the numerical analysis of an equilibrium in the economy with an illiquid stock market and provide their interpretation. We choose the following calibration of the model unless pointed otherwise: $\gamma_2 = \gamma_3 = 1$, $\varepsilon = 1$, $T = 0.5$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $\lambda_1 = 0.49$, and $\lambda_2 = 0.01$. We set T to be rather small because we are interested in a stock market that is illiquid. We study an economy where the stock market is significantly illiquid by setting α to 0.2. We will discuss the changes in our results if α decreases. Furthermore, we find the equilibrium in N_3X_1–space and show the equilibrium processes when time is zero and one of the state variables is fixed. For convenience, we show the results versus total holdings of investors, $N_i = 1 - (1 - \lambda_1 - \lambda_2)N_3$ rather than N_3. For the same reason, we depict the rate of trading of investors, $u_i = -(1 - \lambda_1 - \lambda_2)u_3$, rather than u_3.

A very small weight of investors of the second type results in their impact on the stock prices being negligible unless investors of the first type have extremely high risk exposure, $|N_1|/X_1$. This observation often allows us to neglect the difference between the aggregate investor 1 and the aggregate investor representing all investors.

4.1 Strategies and Returns

Panel A of Figure 2 depicts the conditional stock market premium and the volatility of the stock return (the thick solid and dashed lines, respectively), the conditional Sharpe ratio (the thin solid line) as well as the rate of trading of investors u_i (the thin dashed line) as functions of N_i when X_1 is fixed. The panel shows that the conditional risk premium and the Sharpe ratio increase monotonically with N_i from very negative values when N_i is small to very large values when N_i is large. Moreover, the rate of trading between agents decreases from being positive to negative as N_i grows from small to large values. If N_i is large, then investors have an excessive risk exposure and are willing to bear the risk only if the conditional risk premium and the Sharpe ratio are high. On the other end of trading, market makers have short or small positions in the stock market and would like to buy shares. Therefore, the overall direction of the stock trading is buy for market makers and sell for the aggregate investor. Moreover, because market makers have to pay significant transaction costs for their trading, they are willing to have a small position in the stock market when the conditional risk premium and the Sharpe ratio are high. Naturally, the higher N_i and the smaller positions of market makers, the faster they trade to leave these unfavorable states. Similar, if N_i is negative, then investors have excessively risky liabilities and are willing to keep them only if the conditional risk premium as well as the Sharpe ratio are negative. On the other hand, market makers have excessively long positions in the stock market and want to sell their stock shares. Therefore, the overall direction of the stock trading is sell for market makers and buy for the aggregate investor. Furthermore, market makers tolerate their long positions in the stock with a negative risk premium since adjustments of their stock allocations are very costly to them.

Given the above behavior of the conditional stock returns, let us look at the trading of the aggregate investor in more detail. Panel A of Figure 2 shows that if N_i is positive and increases (or negative and falls) then the risk premium rises (falls) so that stock price decreases (increases). Therefore, investors have an incentive to buy cheap stock shares when they are long or take shorter position when they are short. This is the substitution effect. On the other hand, the aggregate investor looses money on his current position in the stock market when N_i is positive and increases since he purchased shares at a price higher than today. Similarly, the aggregate investor looses money when N_i is negative and decreases since he had a short position in the stock when it was cheaper than today. The resulting losses cause the wealth effect that makes the aggregate investor sell his stock shares when N_i is positive or buy them when N_i is negative. Moreover, the higher N_i in magnitude, the faster investors want to trade. Because investors sell (buy) stock shares when N_i is significantly positive (negative), we conclude that the wealth effect plays a dominant role for them at these states of the economy. Also, if investors sell shares when N_i is significantly positive, they put a pressure on the stock price to decrease and the conditional Sharpe ratio to increase. Similarly, if N_i is small, then investors buy stock shares and exert a pressure on the stock price to increase and the conditional Sharpe ratio to decrease. Because the described behavior of returns is seen on Panel A of Figure 2, we conclude that the wealth effect of investors has a leading influence on the stock returns when N_i is substantially positive or negative. As in the liquid market, the wealth effect plays a leading role for the trading strategies of the aggregate investor. However, while in the liquid market the resulting impact on the stock prices is offset by the substitution effect from market makers, in the illiquid market the role of market makers in price making is substantially weakened by the transaction costs that they face and the wealth effect from investors dominates the price formation in the stock market.

It is interesting that the illiquid economy allows states where the conditional premium for holding the stock shares is negative. As was shown in Section 3, such a situation is not possible in the liquid economy: No matter what the calibration of an economy is, the stock market premium is always positive. That is, even though the fundamentals suggest that the conditional premium for holding stock shares should be positive, it could be negative in the illiquid economy. These conclusions could be surprising since one would normally expect that in the illiquid economy where market makers have to pay transaction costs, the conditional premium for holding the stock market should be above that in the liquid economy.

Panel A of Figure 2 shows that the stock market becomes very volatile in states where investors have very long or very short positions in the stock. The volatility rises because of the wealth effect on investors: If the economy is in a state with a substantially positive (negative) stock market premium and this premium rises (falls) so that the stock price decreases (increases), then investors sell (buy) more stock shares to make the stock price decrease (increase) further. This strategy causes the stock return volatility to increase. This behavior of the stock return volatility is quite different from the one in the liquid economy

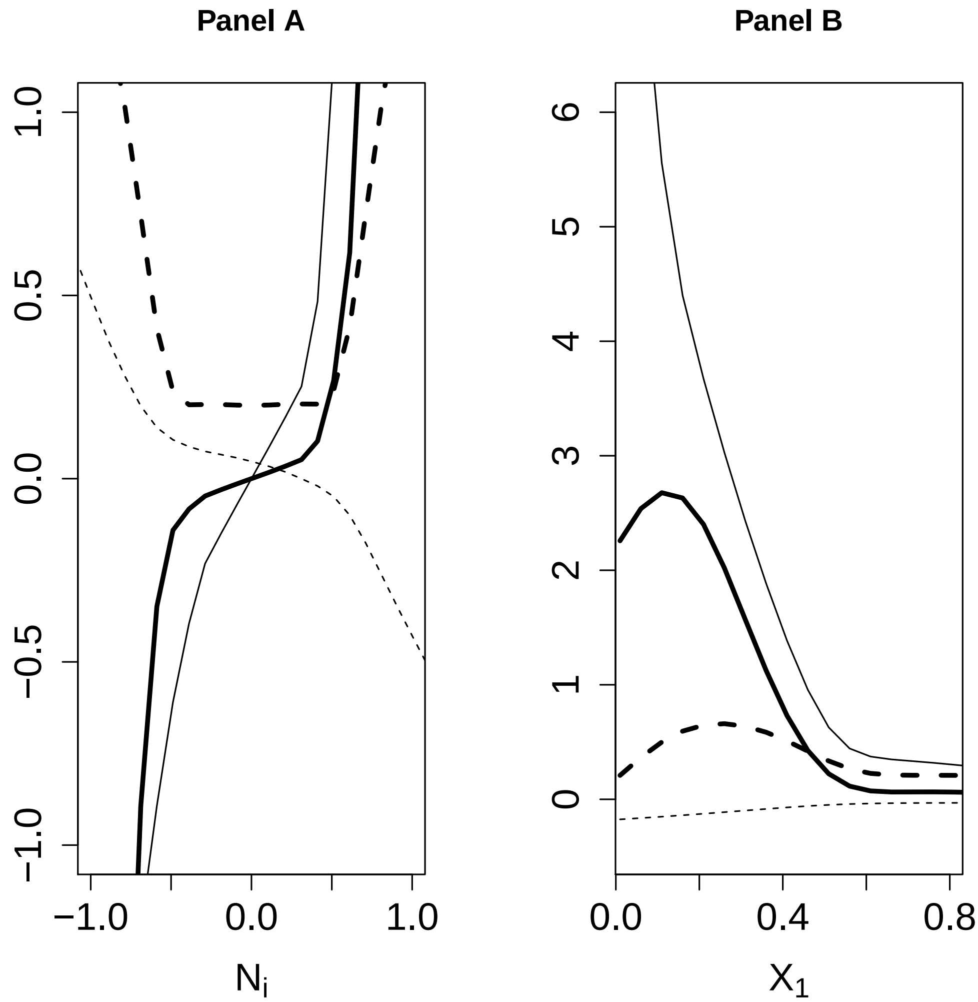

Figure 2: This Figure shows the stock market premium, μ_S (the thick solid line), the volatility of the stock returns, σ_S (the thick dashed line), the conditional Sharpe ratio (the thin solid line), and the rate of trading by investors, u_i (the thin dashed line), against the stock holdings by investors, N_i (Panel A), and against the wealth of the aggregate investor 1, X_1 (Panel B). Panel A assumes that $X_1 = 0.5$ while Panel B assumes that $N_i = 0.5$. The other parameters are set at $\alpha = 0.2$, $\varepsilon = 1$, $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $T = 0.5$, $\lambda_1 = 0.49$, $\lambda_2 = 0.01$, and $t = 0$.

where the upward pressure on the stock return volatility from logarithmic investors is fully offset by the downward pressure from market makers.

Based on the above analysis, we present a scenario related to a financial crisis when the stock market becomes significantly illiquid. As the stock market becomes illiquid, investors want to reduce their stock holdings due to its illiquidity (e.g., see Isaenko, 2010) and more pessimistic expectations on stock returns. Therefore, investors in the illiquid regime of economy start from excessively long allocations to the stock market (so that N_i is high). At these allocations, investors sell their shares and the expected risk premium and Sharpe ratio increase substantially above those in the liquid market. Moreover, the excessive leverage of investors feeds a significant increase in the stock return volatility. On the other end of trading, market makers face substantial losses in trying accommodate a selling pressure from investors. Ex ante, they are partially compensated by buying stock shares that have very high expected Sharpe ratio. Unfortunately, the realized returns of the stock market turned out to be negative on many occasions causing further losses of market makers. These strategies by investors and market makers as well as the expected stock returns are historically observed during dramatic financial distresses such as those in 1929, 1987, 2008 and others when the stock market liquidity plunged. We show that these strategies and stock returns are feasible in a general equilibrium and do not follow from irrational behavior.

Panel B of Figure 2 shows the stock market premium and the volatility of the stock return (the thick solid and dashed lines, respectively), the conditional Sharpe ratio (the thin solid line) as well as the rate of trading u_i (the thin dashed line) versus X_1 at fixed positive N_i. If the aggregate investor 1 becomes poor, his absolute risk aversion increases making him reduce his risk exposure N_1. Therefore, the stock market is cleared by exponential investors who are willing to take a long position in the stock market only if the conditional Sharpe ratio is large. On the other side of the trading, market makers have low risk exposure when the conditional Sharpe ratio is high, so that market makers buy at high speed which increases further with decreasing X_1. The stock return volatility is relatively small in these states because the impact of the wealth effect from the aggregate investor 1 is negligible. Moreover, the stock market premium subsides with smaller X_1 due to the wealth effect but still remains very high due an excessive leverage of exponential investors.

The impact of the wealth effect rises as we increase X_1 from small values and becomes maximal when the aggregate investor 1 is neither too rich nor too poor, and then subsides as X_1 increases further. The wealth of the aggregate investor 1 at which the wealth effect is maximal increases with the growing stock position of investors, N_i. Moreover, the maximal strength of this effect increases with the size of this position as well. Furthermore, as the volatility of the stock return changes with the rising wealth of the aggregate investor, so does the stock market premium to ensure a proper demand for stock shares. The analysis of strategies and returns versus X_1 for a fixed negative N_i is a straightforward extension of the one we carry for a fixed positive N_i.

The cross–section in N_i–dimension is important for understanding an economical intuition behind the stock market mispricing while the cross–section in X_1–dimension provides an insight into the dynamics of stock returns. Indeed, if the stock market illiquidity is high then time variation of the economy is propelled by idiosyncratic shocks to the wealth of the aggregate investor.

4.2 Predictability

In this subsection we consider the short–term predictability of the stock market returns. This predictability is driven by the variations of the risk premium in X_1–dimension. Autocorrelation in returns of an illiquid stock has been considered in a few asset pricing models.[16] These models predict a negative sign for the coefficient of autocorrelation implying that a random increase of the illiquid stock return today is likely to result in a decrease of this return tomorrow. We will show that the autocorrelation coefficient is negative in many states of economy and could be very large in magnitude. Moreover, in the states of economy where the risk exposure of investors is very high, the autocorrelation in stock returns is positive and significant. The latter behavior of the autocorrelation coefficient is very different from that in the liquid economy where it is always negative and negligibly small. We consider only a short term autocorrelation coefficient defined as

$$\begin{aligned} Corr_{\Delta S}(t) &= \frac{Cov_t(\Delta S_t, \Delta S_{t+\Delta t})}{\sqrt{Var_t(\Delta S_t) Var_t(\Delta S_{t+\Delta t})}} \\ &= \frac{sgn(\mu_S)\, \Psi\, \Delta t}{\sqrt{[\sigma_D \sigma_S/(X_1 \mu_S) + h_{X_1} + \Phi \Delta t]^2 + \Upsilon^2 \Delta t + \Psi^2 (\Delta t)^2}}, \end{aligned} \tag{22}$$

where $\Delta S_t = S(t + \Delta t) - S(t)$, Δt is very small, the expectations are conditional on the information at time t, while functions $\Upsilon(t, X_1, N_3)$, $\Psi(t, X_1, N_3)$ and $\Phi(t, X_1, N_3)$ are defined in Appendix F.

Panel A of Figure 3 shows the instantaneous autocorrelation coefficient of the stock returns versus N_i assuming that $\Delta t = 0.02$ and X_1 is fixed. The autocorrelation coefficient of the stock returns in the illiquid market is negative unless the aggregate investor has excessively leveraged or short positions. The coefficient is small in magnitude if the risk exposure of the aggregate investor is small. However, the coefficient becomes highly significant with an increasing stock position and reaches the minimal value of -90% at allocations where the wealth effect is strong. In these states, if the aggregate investor is very long (short) then a random increase in the stock price causes the wealth of the aggregate investor to rise (fall). Since the marginal risk premium with respect to X_1 in these states is negative (positive), the rise (fall) of the wealth causes the stock drift to fall. A smaller drift today results in a smaller stock price tomorrow.

It is interesting that excessive leveraged or short positions of the aggregate investor result in a significantly positive autocorrelation. If the aggregate investor is very long (short) then a random increase in the stock price causes the wealth of the aggregate investor to rise (fall).

[16] For example, see Ho and Stoll (1981), Grossman and Miller (1988), Huang and Wang (2009) and others.

Since the marginal risk premium with respect to X_1 is positive (negative) in these states, the rise (fall) of the wealth makes the stock market premium increase. A higher premium today results in a higher stock price tomorrow.

Panel B of Figure 3 shows the instantaneous autocorrelation coefficient of the stock returns versus X_1 when $N_i = 0.5$. In agreement with our previous discussion, the autocorrelation coefficient is small and negative when X_1 is large and the wealth effect is small. This coefficient follows the wealth effect with decreasing X_1 and approaches the minimal value of -50% when this effect is maximal. As the wealth of the aggregate investor 1 decreases further and the wealth effect weakens, the autocorrelation coefficient falls in magnitude and then becomes large and positive when the aggregate investor is poor and has high leverage. In the latter states, a positive shock to the stock price causes the wealth of the investor to fall which in turn makes the premium increase. A higher premium implies a higher stock price tomorrow and an increase of the risk exposure of the aggregate investor 1. This investor avoids an escalation of the risk in his portfolio through the risk sharing with exponential investors.

4.3 Comparative Statics

In this subsection we consider the changes in the equilibrium resulting from variations in the main parameters of the economy defining its liquidity: α, the weight of investors in economy, $\lambda_1 + \lambda_2$, and T. We analyze the behavior of the conditional moments of the stock returns and the rate of trading between agents. Extrapolation of our conclusions to the predictability of the stock returns is intuitive and will not be discussed.

Figure 4, Panels A and B replot the conditional moments of the stock market returns shown in Figure 2 for a more liquid stock market in which coefficient α is equal to 0.04 rather than 0.2 while all other parameters remaining unchanged. We do not show the rate of trading for clarity of exposition. Smaller α implies faster trading by market makers. As market makers trade faster, the time that the economy stays in the states with a strong wealth effect decreases, making the impact of this effect on the prices fall at the given allocation of investors, N_i. Hence, the stock market premium and the volatility of stock returns decrease in magnitudes if α subsides. In the limit of a very small α where the stock market is almost liquid, the wealth effect becomes negligible for a very wide range of N_i, so that the prices are very close to those in the liquid market except for the states with a very high exposure to the stock market risk by the aggregate investor, in which the wealth effect is still significant and the stock mispricing cannot be neglected.

Figure 5, Panels A and B repeat the conditional moments of the stock market returns shown in Figure 2 for $\lambda_1 = 0.39$, $\lambda_2 = 0.01$ while keeping all other parameters the same. We compare the stock returns in the two economies at the same aggregate allocations of investors, $N_i = 1 - (1 - \lambda_1 - \lambda_2)N_3$. Fewer investors results in decreasing the stock market liquidity. Consequently, the wealth effect and the mispricing increases. Figure 5 shows that even a small decrease of the proportion of investors in economy results in a significant

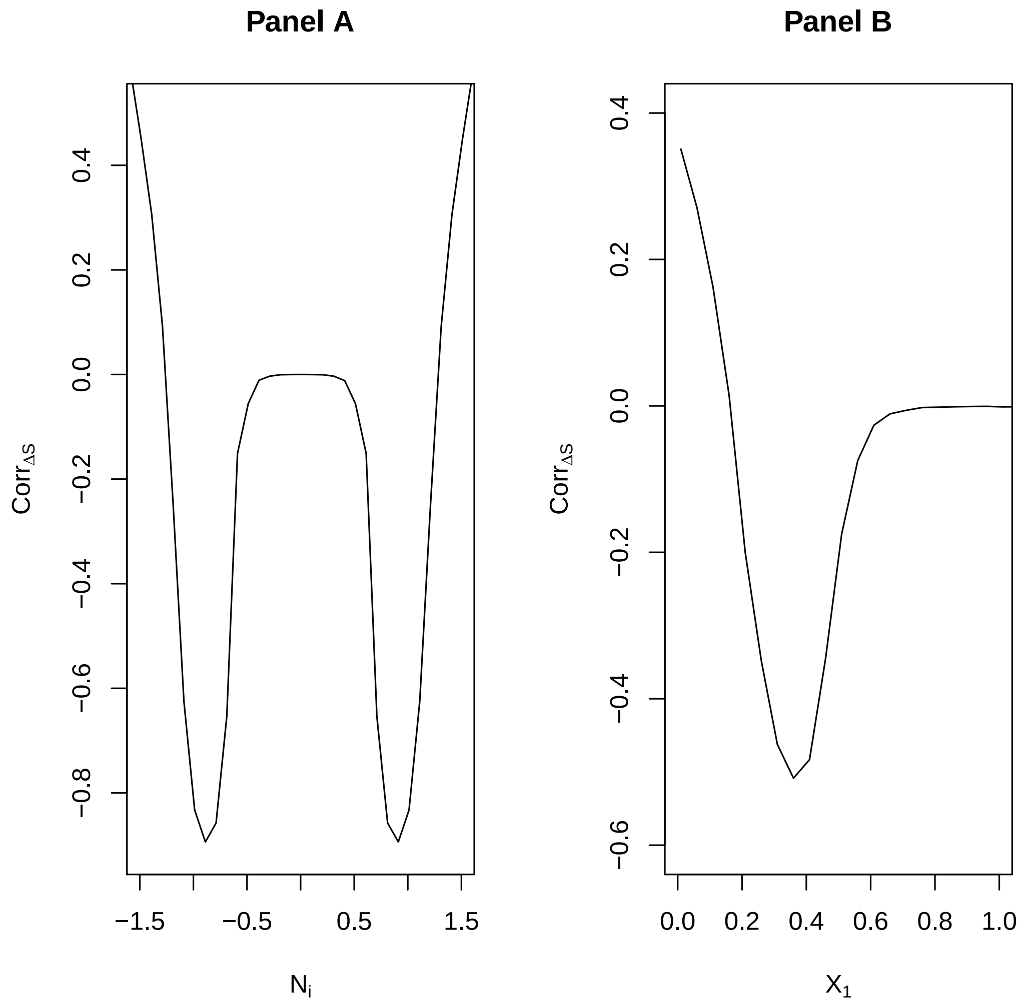

Figure 3: This Figure shows the autocorrelation coefficient, $Corr_{\Delta S}$, against the stock holdings by market makers, N_i (Panel A), and against the wealth of the aggregate market maker, X_1 (Panel B). Panel A assumes that $X_1 = 0.5$ while Panel B depicts that $N_i = 0.5$. The other parameters are set at $\alpha = 0.2$, $\varepsilon = 1$, $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $T = 0.5$, $\lambda_1 = 0.49$, $\lambda_2 = 0.01$, and $t = 0$.

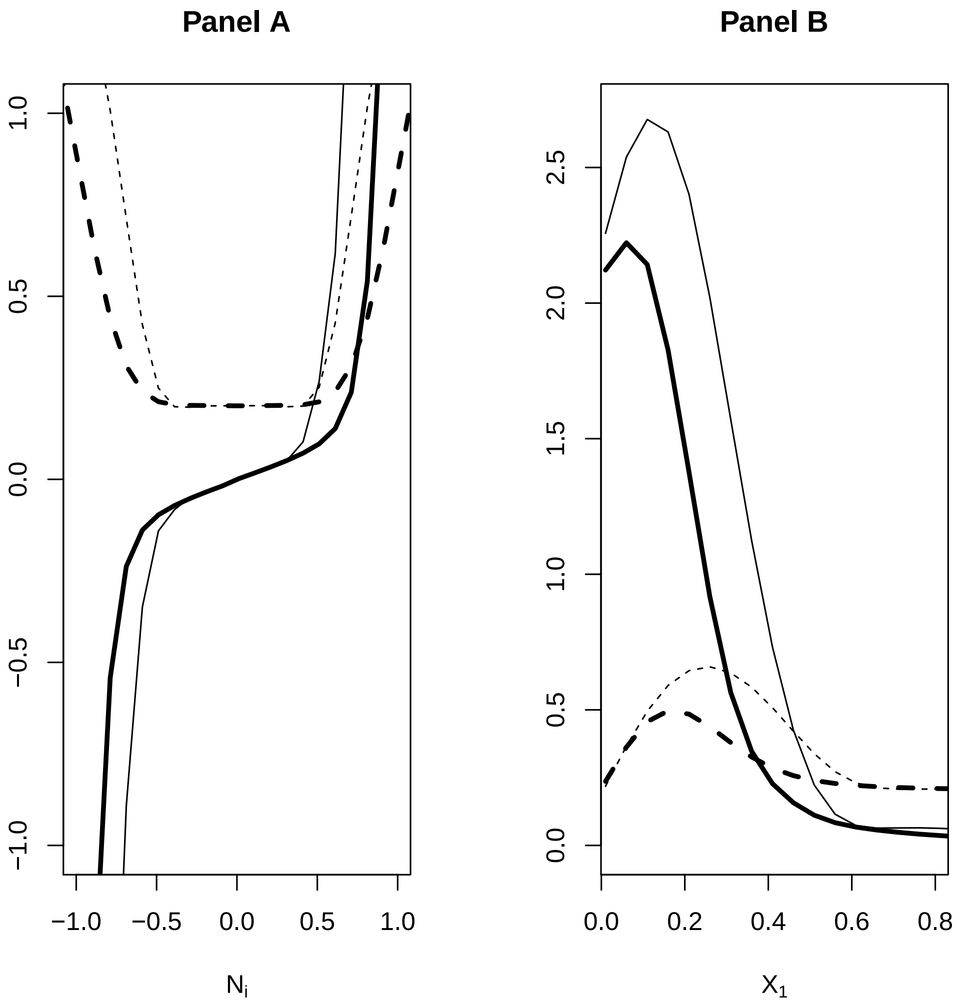

Figure 4: This Figure shows the conditional premium on the stock market, μ_S (the solid lines) and the volatility of the stock returns, σ_S (the dashed lines) against the stock holdings of market makers, N_i (Panel A) and against the wealth of the aggregate investor 1, X_1 (Panel B). The thin lines correspond to $\alpha = 0.2$ while the thick lines assume $\alpha = 0.04$. Panel A assumes that $X_1 = 0.5$ while Panel B assumes that $N_i = 0.5$. The other parameters are set at $\varepsilon = 1$, $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $T = 0.5$, $\lambda_1 = 0.49$, $\lambda_2 = 0.01$, and $t = 0$.

increase of the stock market mispricing. Furthermore, decreasing proportion of investors makes them trade much faster in a given state of the economy X_1, N_i.

Panels A and B of Figure 6 repeat the conditional moments of the stock market returns shown in Figure 2 when $T = 0.25$, while keeping all other parameters unchanged. If the investment horizon of investors increases then returns from their stock holdings become more risky. Therefore, the stock market clears only if the risk premium increases where the aggregate investor is long and decreases where he is short so that the stock market mispricing increases. Furthermore, potential losses of market makers increase in the states with high $|N_i|$ making them trade faster with a longer investment horizon. Figure 6 shows that the wealth effect and the stock mispricing subsides relatively quickly over time. The impact from the diminishing wealth effect is partly offset by the augmented time that an economy spends in the states with significant mispricing.

5 Policy Interventions

Based on the results of comparative statics, one can suggest possible market adjustments aimed to decrease mispricing and mitigate financial crises. The mitigation could be achieved by means of facilitating trading for market makers leading to a decrease of wealth effect. The latter could be done in the two major directions: Making market microstructure more efficient so that the coefficient in the transaction costs, α, decreases; policy interventions. The former could be done by improving search efficiency, the inventory–absorbtion capacity and competition among market makers. The latter could be achieved in the two different ways: decrease of the interest rate and direct asset purchase. Below we briefly discuss these two policies.

Even though we use a locally riskless bond as a numeraire, one can predict what should happen to the stock mispricing if the interest rate suddenly changes. During financial crises a market maker buys stock and sells liquid T-bills. Therefore, the government can buy T-bills to increase their price and the returns of market makers. Alternatively, the government can decrease the short–term interest rate until borrowing at this rate becomes cheaper than lending at T-bill rate.

The government can buy risky securities from investors and decrease their leverage. The latter will reduce the pressure from market makers to provide liquidity and will result in a smaller expected volatility of the stock market. The government can finance its purchase by raising taxes to households or issuing T-bills. The latter, however, will downside the profits of market makers who are selling T-bills and therefore is not recommended.

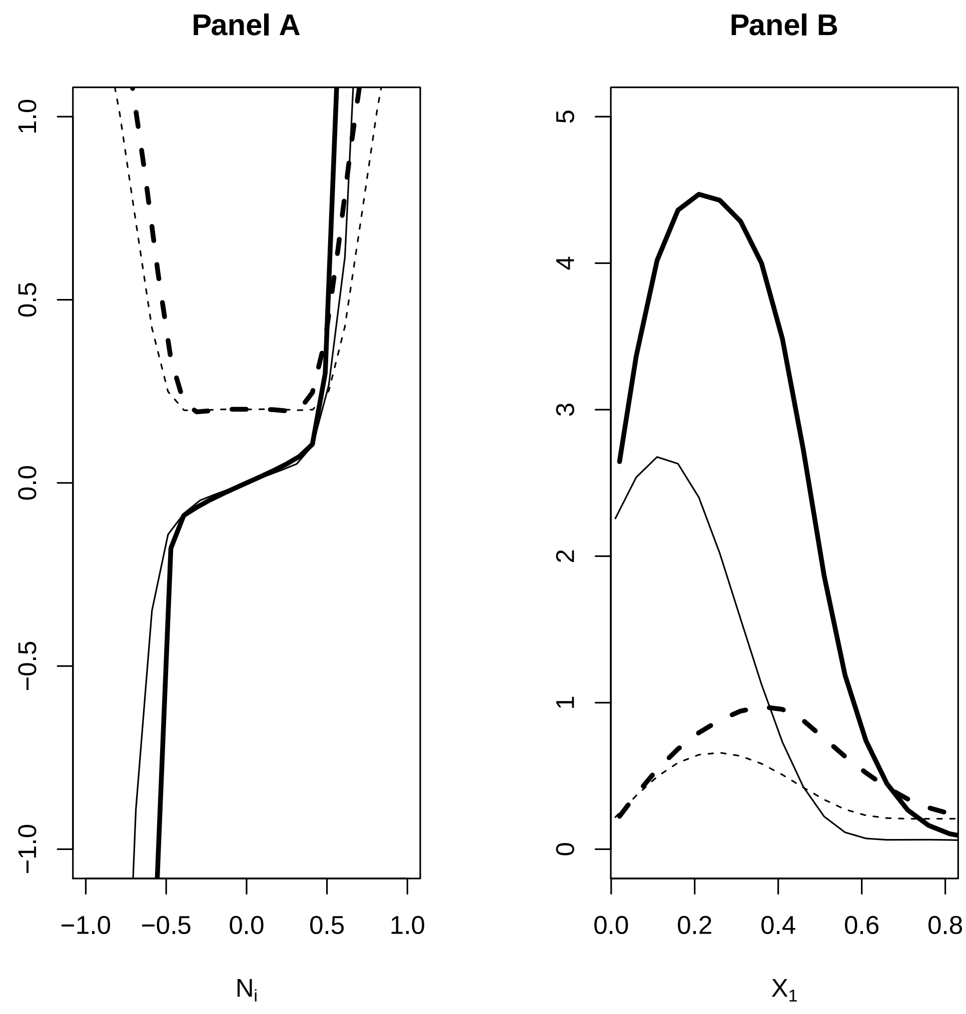

Figure 5: This Figure shows the conditional premium on the stock market, μ_S (the solid lines) and the volatility of the stock returns, σ_S (the dashed lines) against the stock holdings by investors, N_i (Panel A) and against the wealth of the aggregate investor 1, X_1 (Panel B). The thin lines correspond to $\lambda_1 = 0.49$, $\lambda_2 = 0.01$ while he thick lines assume $\lambda_1 = 0.39$, $\lambda_2 = 0.01$. Panel A assumes that $X_1 = 0.5$ while Panel B assumes that $N_i = 0.5$. The other parameters are set at $\varepsilon = 1$, $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $T = 0.5$, and $\alpha = 0.2$.

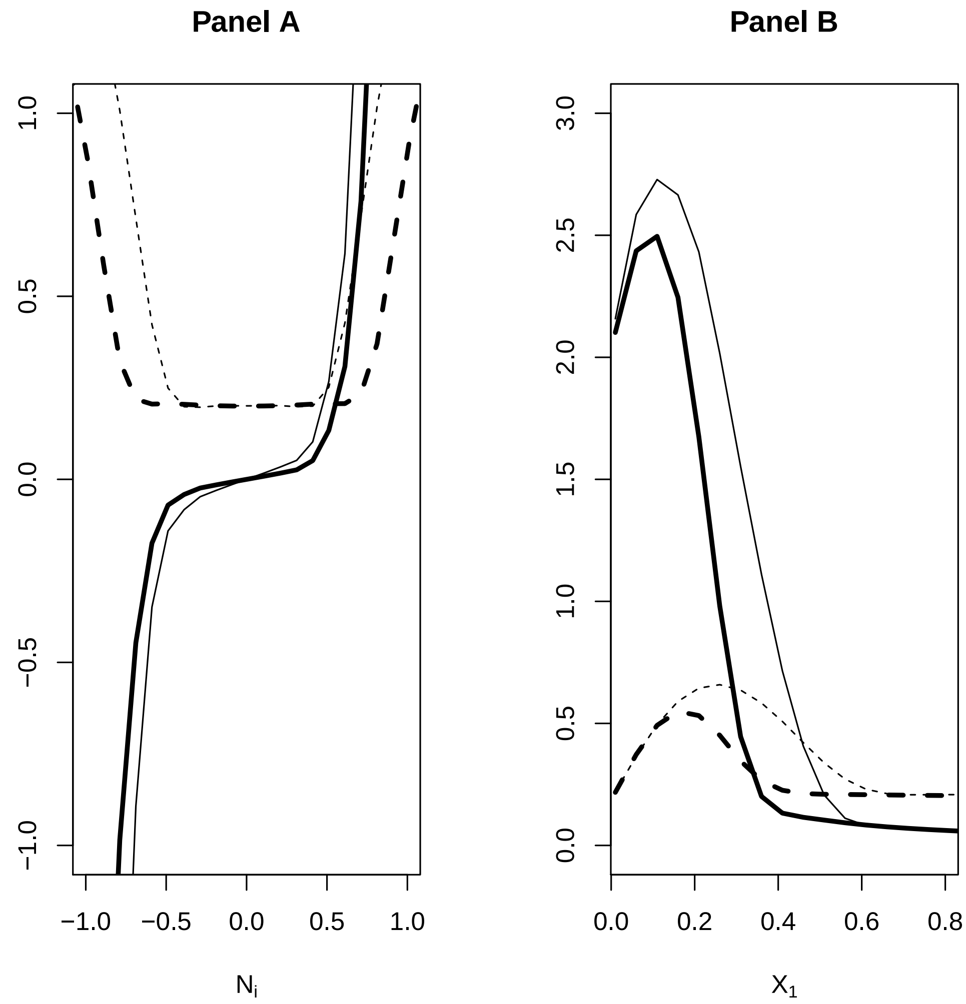

Figure 6: This Figure shows the conditional premium on the stock market, μ_S (the solid lines) and the volatility of the stock returns, σ_S (the dashed lines) against the stock holdings by investors, N_i (Panel A) and against the wealth of the aggregate market maker, X_1 (Panel B). The thin lines correspond to $T = 0.5$ while he thick lines assume $T = 0.25$. Panel A assumes that $X_1 = 0.5$ while Panel B assumes that $N_i = 0.5$. The other parameters are set at $\alpha = 0.2$, $\varepsilon = 1$, $\gamma_2 = \gamma_3 = 1$, $\mu_D = 0.07$, $\sigma_D = 0.2$, $\lambda_1 = 0.49$, $\lambda_2 = 0.01$, and $t = 0$.

6 Conclusion

In this paper we find a dynamic equilibrium in an economy with an illiquid stock market where investors trade with market makers who face significant trading losses that are modeled by means of transaction costs that are convex in the number of traded shares. Our model is able to explain the increased volatility, very large positive expected stock returns and the conditional Sharpe ratios observed during financial crises. We also find that the short–term autocorrelation of stock returns in an illiquid market is much bigger than it is in the liquid market and can be positive, while the autocorrelation coefficient in the liquid market is always negative.

Appendix A

Let $Z = (X_1, N_3)^T$ be a vector of macro state variables. We solve the problem faced by an investor 1 by using the Hamilton–Jacobi–Bellman (HJB) equation for his value function $V_1^i(t, X_1^i, Z)$:

$$\begin{aligned}
&\max_{N_1^i \in R} \Bigg\{ V_{1t}^i + \frac{1}{2}\Big(N_1^i \sigma_S\Big)^2 V_{1X_1^i X_1^i}^i + \frac{1}{2}\Big(N_1 \sigma_S\Big)^2 V_{1X_1X_1}^i + N_1^i \sigma_S^2 N_1 V_{1X_1^i X_1}^i + N_1^i \mu_S V_{1X_1^i}^i \\
+ \quad & u_3 V_{N_3}^i + N_1 \mu_S V_{1X_1}^i \Bigg\} = 0, \quad V_1^i(T, X_1^i, Z) = \ln X_1^i, \qquad \text{(A-1)}
\end{aligned}$$

where N_1 is a number of shares hold by an aggregate investor 1. It follows that

$$N_1^i = -\frac{\mu_S V_{1X_1^i}^i + \sigma_S^2 N_1 V_{1X_1^i X_1}^i}{\sigma_S^2 V_{1X_1^i X_1^i}^i}.$$

We conjecture that $V_1^i = \ln X_1^i + g_1(t, Z)$, then

$$N_1^i = X_1^i \frac{\mu_S}{\sigma_S^2} \qquad \text{(A-2)}$$

and $g_1(t, Z)$ solves the following PDE

$$g_{1t} + \frac{1}{2}\left(\frac{\mu_S}{\sigma_S}\right)^2 + \frac{1}{2}\Big(N_1 \sigma_S\Big)^2 g_{1X_1X_1} + u_3 g_{1N_3} + N_1 \mu_S g_{1X_1} = 0, \quad g_1(T, Z) = 0. \qquad \text{(A-3)}$$

In equilibrium $N_1 = X_1 \frac{\mu_S}{\sigma_S^2}$. Therefore the last PDE can be rewritten as

$$g_{1t} + \left(\frac{\mu_S}{\sigma_S}\right)^2 \left(\frac{1}{2} + \frac{1}{2} X_1^2 g_{1X_1X_1} + X_1 g_{1X_1}\right) + u_3 g_{1N_3} = 0, \quad g_1(T, Z) = 0. \qquad \text{(A-4)}$$

The Hamilton–Jacobi–Bellman (HJB) equation for the value function of an investor 2, $V_2^l(t, X_2^l, Z)$, is

$$\begin{aligned}
&\max_{N_2^l \in R} \Bigg\{ V_{2t}^l + \frac{1}{2}\Big(N_2^l \sigma_S\Big)^2 V_{2X_2^l X_2^l}^l + \frac{1}{2}\Big(N_1 \sigma_S\Big)^2 V_{2X_1X_1}^l + N_2^l \sigma_S^2 N_1 V_{2X_2^l X_1}^l + N_2^l \mu_S V_{2X_2^l}^l \\
+ \quad & u_3 V_{N_3}^l + N_1 \mu_S V_{2X_1}^l \Bigg\} = 0, \quad V_2^l(T, X_1^l, Z) = -\frac{1}{\gamma_2} e^{-\gamma_2 X_2^l}. \qquad \text{(A-5)}
\end{aligned}$$

It follows that

$$N_2^l = -\frac{\mu_S V_{2X^l}^l + \sigma_S^2 N_1 V_{2X_2^l X_1}^l}{\sigma_S^2 V_{2X_2^l X_2^l}^l}.$$

We conjecture that $V_2^l = -\frac{1}{\gamma_2} e^{-\gamma_2 X_2^l} g_2(t, Z)$, then

$$N_2^l = \frac{\mu_S}{\gamma_2 \sigma_S^2} + \frac{N_1 g_{2X_1}}{\gamma_2 g_2} = \frac{\mu_S}{\gamma_2 \sigma_S^2}\left(1 + \frac{X_1 g_{2X_1}}{g_2}\right) \qquad \text{(A-6)}$$

and $g_2(t,Z)$ solves the following PDE

$$
\begin{aligned}
0 &= g_{2t} - \frac{1}{2}\left(\frac{\mu_S}{\sigma_S} + \frac{N_1\sigma_S g_{2X_1}}{g_2}\right)^2 g_2 + \frac{1}{2}(\sigma_S N_1)^2 g_{2X_1X_1} + u_3 g_{2N_3} + N_1\mu_S g_{2X_1}, \\
& g_2(T,Z) = 1. \qquad \text{(A-7)}
\end{aligned}
$$

In equilibrium the last PDE can be rewritten as

$$
0 = g_{2t} + \left(\frac{\mu_S}{\sigma_S}\right)^2 \left[X_1 g_{2X_1} + \frac{1}{2}X_1^2 g_{2X_1X_1} - \frac{1}{2}\left(1 + \frac{X_1 g_{2X_1}}{g_2}\right)^2 g_2\right] + u_3 g_{2N_3}. \qquad \text{(A-8)}
$$

In equilibrium the Sharpe ratio depends on g_2. We find this ratio by using the market clearing condition.

Appendix B

The indirect utility function of a market maker $V_3^m(t, X_3^m, X_1, N_3)$ solves

$$
\begin{aligned}
0 &= \max_{u_3\in R}\Big\{ V_{3t}^m + \frac{1}{2}(\sigma_S N_3)^2 V_{3X_3^mX_3^m}^m + \frac{1}{2}(\sigma_S N_1)^2 V_{3X_1X_1}^m + \sigma_S^2 N_3 N_1 V_{3X_3^mX_1}^m + u_3 V_{3N_3}^m \\
&+ (N_3\mu_S - \alpha|u_3|^{1+\varepsilon})V_{3X_3^m}^m + N_1\mu_S V_{3X_1}^m \Big\}, \quad V_3^m(T, X_3^m, X_1, N_3) = -\frac{1}{\gamma_3}\exp(-\gamma_3 X_3^m).
\end{aligned}
$$

We assume that $V_3^m(t, X_3^m, X_1, N_3) = -\frac{1}{\gamma_3}\exp(-\gamma_3 X_3^m) f(t, X_1, N_3)$ and find the PDE for f:

$$
\begin{aligned}
0 &= \min_{u_3\in R}\Big\{ f_t + \frac{1}{2}(\gamma_3\sigma_S N_3)^2 f + \frac{1}{2}(\sigma_S N_1)^2 f_{X_1X_1} - \gamma_3\sigma_S^2 N_1 N_3 f_{X_1} + u_3 f_{N_3} \\
&- \gamma_3(N_3\mu_S - \alpha|u_3|^{1+\varepsilon})f + N_1\mu_S f_{X_1}\Big\}, \quad f(T, X_1, N_3) = 1. \qquad \text{(B-1)}
\end{aligned}
$$

The first order condition provides the optimal rate of trading given by

$$
u_3 = -sgn(f_{N_3})\left(\frac{|f_{N_3}|}{\alpha(1+\varepsilon)\gamma_3 f}\right)^{\frac{1}{\varepsilon}}. \qquad \text{(B-2)}
$$

Substitution of the last result into equation (B-1) gives the PDE for $f(t, X_1, N_3)$:

$$
\begin{aligned}
& f_t + \left[\frac{1}{2}(\gamma_3\sigma_S N_3)^2 - \gamma_3 N_3\mu_S\right] f + \frac{1}{2}\left(\frac{\mu_S}{\sigma_S}X_1\right)^2 f_{X_1X_1} + X_1\left(\frac{\mu_S}{\sigma_S^2}\right)[\mu_S - \gamma_3\sigma_S^2 N_3] f_{X_1} \\
&- \varepsilon\left(\frac{|f_{N_3}|/(\gamma_3 f)}{\alpha(1+\varepsilon)^{1+\varepsilon}}\right)^{\frac{1}{\varepsilon}} |f_{N_3}| = 0, \qquad f(T, X_1, N_3) = 1,
\end{aligned}
$$

where we replaced N_1 with $X_1\mu_S/\sigma_S^2$. The last equation depends on μ_S and σ_S which will be found endogenously as functions of t, N_3, and X_1.

Appendix C

In this Appendix we derive the PDE for the stock price in an illiquid economy. First, we find μ_S and σ_S. The stock market clearing condition reads

$$\frac{\mu_S}{\sigma_S^2}\left[\lambda_1 X_1 + \lambda_2\left(\frac{1}{\gamma_2} + \frac{X_1 g_{2X_1}}{\gamma_2 g_2}\right)\right] + (1-\lambda_1-\lambda_2)N_3 = 1.$$

Therefore,

$$\mu_S = \sigma_S^2 \Omega, \tag{C-1}$$

where

$$\Omega = \frac{1-(1-\lambda_1-\lambda_2)N_3}{\left[\lambda_1 X_1 + \lambda_2\left(\frac{1}{\gamma_2} + \frac{X_1 g_{2X_1}}{\gamma_2 g_2}\right)\right]}.$$

Moreover, because the set of state variables for the stock price in this economy is D, X_1 and N_3, we find from the Itô formula that $\sigma_S = \sigma_D S_D + \sigma_S N_1 S_{X_1}$, or

$$\sigma_S = \frac{S_D \sigma_D}{1-\Omega S_{X_1} X_1}. \tag{C-2}$$

Let ξ_t be the pricing kernel in our economy with dynamics

$$d\xi_t = -\xi_t\left(\frac{\mu_{St}}{\sigma_{St}}\right)dW_t, \quad \xi_0 = 1. \tag{C-3}$$

Since the process $\xi_t S_t$ is a martingale, its drift is zero. Therefore, $S_t = \frac{1}{\xi_t}E_t[\xi_T D_T]$ and we find the PDE for S:

$$\begin{aligned} 0 \;&=\; S_t + (\mu_D - \frac{\mu_S}{\sigma_S}\,\sigma_D)\,S_D + u_3 S_{N_3} + \frac{1}{2}\sigma_D^2\,S_{DD} + \frac{1}{2}N_1^2\sigma_S^2\,S_{X_1X_1} + \sigma_D\sigma_S N_1\,S_{DX_1}, \\ & \quad S(T,D,X_1,N_3) = D. \end{aligned}$$

Let us conjecture that $S(t,D,X_1,N_3) = D + h(t,X_1,N_3)$ then the last PDE results in

$$0 \;=\; h_t + \mu_D - \frac{\mu_S}{\sigma_S}\,\sigma_D + u_3 h_{N_3} + \frac{1}{2}\left(\frac{\mu_S}{\sigma_S}X_1\right)^2 h_{X_1X_1}, \quad h(T,X_1,N_3) = 0, \tag{C-4}$$

where $\frac{\mu_S}{\sigma_S} = \frac{\Omega\sigma_D}{1-\Omega S_{X_1}X_1}$.

Finally, we verify that u_3 is independent from D. Since σ_S and μ_S are independent from D, we do not have to include D as a state variable in the PDE for V_3. Therefore, u_3 depends on only t, N_3 and X_1.

Appendix D

In this Appendix we describe a computational algorithm for finding an equilibrium in illiquid economy. The equilibrium can be found by simultaneously solving the PDE's for $g_2(t, X_1, N_3)$, $f(t, X_1, N_3)$, and $h(t, X_1, N_3)$. The impact on prices from exponential investors is carried through the term proportional to λ_2 present in the denominator of Ω. Since the weight of these investors is very small, their impact on prices becomes noticeable only at a very small wealth of aggregate market maker 1. If λ_2 is very small then the conditional moments of stock returns can be approximated as

$$\mu_S = \sigma_S^2 \frac{N_i}{\lambda_1 X_1}, \qquad \text{(D-1)}$$

$$\sigma_S = \frac{\sigma_D}{1 - N_i h_{X_1}/\lambda_1}. \qquad \text{(D-2)}$$

If follows that the equilibrium in the states where X_1 is not very small can be approximated by simultaneous solving the PDE's for for f and h and then for g_2. Reduction of the number of equations that has to be solved simultaneously substantially improves the convergence of numerical results. The range of X_1 for which the found predictions do not hold is very small and shrinks proportionally to λ_2. Moreover, we make it negligibly small by commanding the equilibrium to be the one without logarithmic market makers at $X_1 = 0$. Note that after $g_2(t, X_1, N_3)$ is calculated, μ_S and σ_S found from equations (D-1) and (D-2) can be updated by using equations (18) and (19). We check that this update is negligibly small.

We use a finite difference approach and solve the PDE's backwards. We apply Crank–Nicholson scheme for variables X_1 and N_3 in functions f and g_2, and for variable N_3 in function h. We approximate a first order derivative by a left side difference $\frac{F(x)-F(x-\Delta x)}{\Delta x}$ while the second–order derivative in X_1 is approximated by $\frac{F(x+\Delta x)+F(x-\Delta x)-2F(x)}{\Delta x^2}$. We use absorbing boundary conditions for variable X_1 on the boundary $X_1 = X_{max}$ and for variable N_3 on the boundary $N_3 = N_{3min}$. The ranges of the state variables are $X_1 \in [0, 5]$, $N_3 \in [-4, 8]$ while the number of grid points is 100 in X_1–dimension and 100 in N_3–dimensions and 50,000 in t–dimension. We find that the convergence of numerical results deteriorates if α or $\lambda_1 + \lambda_2$ decreases.

Finally, let us find the boundary conditions at $X_1 = 0$. In the limit of a very small X_1, logarithmic investors do not hold stock shares and the equilibrium in the stock market is extrapolated to that without these investors. It is easy to see from Appendix A that the stock holding by exponential investors becomes $N_2 = \frac{\mu_S}{\gamma_2 \sigma_S^2}$ so that the stock clearing condition implies $\mu_S = \gamma_2 N_i \sigma_D^2 / \lambda_2$. Moreover, Appendix C suggests that $\sigma_S = \sigma_D$ and h is a solution of the PDE

$$0 = h_t + \mu_D + h_{N_3} u_3 - \gamma_2 \sigma_D^2 N_i / \lambda_2, \qquad h(T, N_3) = 0, \qquad \text{(D-3)}$$

where u_3 is found from (B-2) and the solution of the PDE for function f

$$0 = f_t + \gamma_3 \left(\frac{\gamma_3}{2} (\sigma_D N_3)^2 - \gamma_2 N_i N_3 \sigma_D^2 / \lambda_2 \right) f - \varepsilon \left(\frac{|f_{N_3}|/(\gamma_3 f)}{\alpha (1+\varepsilon)^{1+\varepsilon}} \right)^{\frac{1}{\varepsilon}} |f_{N_3}| \qquad f(T, N_3) = 1.$$

Similar, in the absence of logarithmic investors g_2 solves the PDE

$$0 = g_{2t} - \frac{1}{2}\left(\frac{\mu_S}{\sigma_S}\right)^2 g_2 + u_3 g_{2N_3}, \quad g_2(T,Z) = 1. \tag{D-4}$$

We solve the last three equations by using an explicit finite difference scheme. The found functions are used as boundary values for h, f and g_2 at $X_1 = 0$.

Appendix E

An equilibrium in the economy with no transaction costs is easy to find by using a social planner. The preferences of the social planner are given by

$$U(D_T) = \sup_{\lambda_1 C_{1T} + (1-\lambda_1)C_{2T} = D_T} z_0 \ln(C_{1T}) - (1-z_0)\frac{e^{-\gamma_3 C_{2T}}}{\gamma_3}, \tag{E-1}$$

where z_0 is a constant depending on the initial endowments of the agents and proportion λ_1[17] and we assume that the coefficient of risk aversion of investor 2 is the same as it is of a market maker.
It follows that the terminal consumption of the aggregate investor, C_{1T}, solves

$$z_0 \frac{1}{C_{1T}} = (1-z_0)\frac{\lambda_1}{1-\lambda_1} e^{-\frac{\gamma_3}{1-\lambda_1}(D_T - \lambda_1 C_{1T})}. \tag{E-2}$$

We find the terminal consumption of market makers from the constraint $\lambda_1 C_{1T} + (1-\lambda_1)C_{2T} = D_T$ and then we obtain $U(D_T)$ from (E-1). The pricing kernel is given by $\xi_t = E_t[\xi_T] = E_t[U'(D_T)]/E_0[U'(D_T)]$. Because $\nu_t = E_t[U'(D_T)]$ is a martingale and D is the only state variable, $\nu(t,D)$ solves:

$$\nu_t + \frac{1}{2}\sigma_D^2 \nu_{DD} + \mu_D \nu_D = 0, \qquad \nu(T,D) = U'(D). \tag{E-3}$$

After solving the last PDE we find $\xi_t = \nu_t/\nu_0$. Because $d\xi_t = -\xi_t(\frac{\mu_{St}}{\sigma_{St}})dW_t$ and $d\nu_t = \nu_D(t,D)\sigma_D\, dW_t$, we derive

$$\frac{\mu_S}{\sigma_S} = -\frac{\nu_D \sigma_D}{\nu}. \tag{E-4}$$

With a help of the last expression, we obtain stock price $S(t,D)$ by solving the following PDE

$$0 = S_t + (\mu_D - \frac{\mu_S}{\sigma_S}\sigma_D)\, S_D + \frac{1}{2}\sigma_D^2\, S_{DD}, \qquad S(T,D) = D. \tag{E-5}$$

[17] If δ_1 and δ_2 are the Lagrange multipliers of agents 1 and 2, respectively, then one can show that $z_0 = \frac{\delta_2\lambda_1}{\delta_2\lambda_1 + \delta_1(1-\lambda_1)}$.

Then, the stock return volatility is found from $\sigma_S = \sigma_D S_D$. To calculate the wealth of investors, $X_2(t, D)$, one has to solve the PDE

$$0 \;=\; X_{2t} + (\mu_D - \frac{\mu_S}{\sigma_S}\,\sigma_D)\,X_{2D} + \frac{1}{2}\sigma_D^2\,X_{2DD}, \qquad X_2(T, D) = C_{2T}, \tag{E-6}$$

while the wealth of the aggregate investor is found from the bond market clearing condition $X_1 = \frac{S-(1-\lambda_1)X_2}{\lambda_1}$. Finally, the allocations to the stock market of the aggregate investor and market makers are given by $N_1 = \frac{\sigma_D X_{1D}}{\sigma_S}$ and $N_2 = \frac{\sigma_D X_{2D}}{\sigma_S}$, respectively.

The equations above can be solved only numerically. Therefore, we first find C_{1T} numerically and obtain the terminal condition for process ν. Then we solve equation (E-3) by using an explicit finite difference approach. Then we solve PDE's (E-5) and (E-6) by using the same approach and the conditional Sharpe ratio found from (E-4).

Finally, we consider an economy in the limit of a large negative D. This economy is dominated by market makers so that the representative agent has utility function at equilibrium given by $-\frac{1}{\gamma_3}\exp[-\frac{\gamma_3 D}{1-\lambda_1}]$. It follows that the pricing kernel in this economy is proportional to $\nu(t, D) = \exp(-\frac{\gamma_3 D}{1-\lambda_1})\exp\{\frac{1}{2}[(\gamma_3\sigma_D/(1-\lambda_1))^2 - \gamma_3\mu_D/(1-\lambda_1)](T-t)\}$ and the Sharpe ratio is equal to $\gamma_3\sigma_D/(1-\lambda_1)$. Solving the PDE for the stock price results in $S(t, D) = D - (\gamma_3\sigma_D^2/(1-\lambda_1) - \mu_D)(T-t)$. Therefore, $\sigma_S = \sigma_D$ and $\mu_S = \gamma_3\sigma_D^2/(1-\lambda_1)$.

Appendix F

In this appendix we derive the coefficients of instantaneous autocorrelations for the stock returns in the liquid and illiquid markets. Assuming that $d\mu_S = \mu_{\mu_S}dt + \sigma_{\mu_S}dW$, $d\sigma_S = \mu_{\sigma_S}dt + \sigma_{\sigma_S}dW$, one can write

$$dS_{t+dt} = \mu_S(t+dt)dt + \sigma_S(t+dt)dW_{t+dt} = \mu_S(t)dt + \mu_{\mu_S}(t)dt^2 + \sigma_{\mu_S}(t)dW_t dt + \sigma_S(t)dW_{t+dt} +$$

$$\mu_{\sigma_S}(t)dt dW_{t+dt} + \sigma_{\sigma_S}(t)dW_{t+dt}dW_t.$$

Because dW_t and dW_{t+dt} are independent, the conditional covariance between two sequential returns is equal to $Cov_t(dS_t, dS_{t+dt}) = \sigma_S(t)\sigma_{\mu_S}(t)dt^2$, while the conditional variances are $var_t(dS_t) = \sigma_S^2(t)dt$, $var_t(dS_{t+dt}) = [(\sigma_S(t) + \mu_{\sigma_S}(t)dt)^2 + \sigma_{\mu_S}^2(t)dt^2 + \sigma_{\sigma_S}^2(t)dt]dt$. It follows that the autocorrelation coefficient of the stock returns can be written as

$$Corr_{\Delta S}(t) = \frac{\sigma_{\mu_S}(t)\Delta t}{\sqrt{[\sigma_S(t) + \mu_{\sigma_S}(t)\Delta t]^2 + \sigma_{\mu_S}^2(t)\Delta t^2 + \sigma_{\sigma_S}^2(t)\Delta t}},$$

where Δt is very small. If the market is liquid then from the Itô's formula $\sigma_S = S_D\sigma_D \approx \sigma_D$, $\mu_S = S_t + S_D\mu_D + \frac{1}{2}S_{DD}\sigma_D^2$. Therefore, $\sigma_{\mu_S} = \mu_{\sigma_S} = \sigma_D(S_{tD} + S_{DD}\mu_D + \frac{1}{2}S_{DDD}\sigma_D^2)$. For the time increment to be used, $\sigma_S \gg |\mu_{\sigma_S}|\Delta t$, $\sigma_S \gg |\sigma_{\mu_S}|\Delta t$ and formula (21) follows.

In the illiquid market, we apply the Itô's formula to $S(t+dt, D, X_1, N_3) = D + h(t + dt, X_1, N_3)$ and obtain $\mu_S(t+dt, X_1, N_3)$, $\sigma_S(t+dt, X_1, N_3)$. Then we reapply the Itô's formula to find σ_{μ_S} and μ_{σ_S}. In particular, we derive $\sigma_{\sigma_S} = X_1 S_p \Upsilon$, $\sigma_{\mu_S} = X_1 S_p \Psi$, $\mu_{\sigma_S} = X_1 S_p \Phi$, where $\Upsilon = h_{X_1X_1} X_1 S_p$, $\Psi = h_{tX_1} + h_{N_3X_1} u_3 + h_{N_3} u_{1X_1} + h_{X_1X_1}[2X_1 S_p^2 + \frac{1}{2}X_1^2 (S_p^2)_{X_1}] + h_{X_1}[S_p^2 + X_1(S_p^2)_{X_1}] + \frac{1}{2}X_1^2 S_p^2 h_{X_1X_1X_1}$ and $\Phi = h_{tX_1} + h_{N_3X_1} u_3 + h_{X_1}(S_p)_t/S_p + h_{X_1}(S_p)_{N_3} u_1/S_p + h_{X_1X_1}[2X_1 S_p^2 + \frac{1}{2}X_1^2 (S_p^2)_{X_1}] + h_{X_1}[S_p^2 + X_1(S_p^2)_{X_1} + \frac{1}{2}X_1^2 S_p (S_p)_{X_1X_1}] + \frac{1}{2}X_1^2 S_p^2 h_{X_1X_1X_1}$ and S_p is a conditional Sharpe ratio.

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